Subproject C04

Hierarchical Galerkin methods for hyperbolic problems with parabolic asymptotics

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TRR 154
Mathematical modeling, simulation, and optimization using the example of gas networks

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Main goals

Asymptotic preserving numerical schemes for hyperbolic problems.

- ▷ hyperbolic balance laws with stiff relaxation
- ▷ convergence to equilibrium and large time behaviour
- ▷ uniformly stable and asymptotic preserving numerical schemes
- ▷ a-posteriori error analysis and sensitivity calculus

Model problem

Gas transport in pipelines is modeled by

$$\partial_t \rho^e + \partial_x q^e = 0, \qquad \text{on } e \in \mathcal{E}, \qquad (1)$$
$$\partial_t q^e + \partial_x p^e = -d^e q^e, \qquad \text{on } e \in \mathcal{E}, \qquad (2)$$

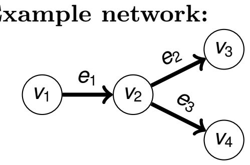
with $d^e = d^e(\rho^e, q^e) = \lambda/(2D)|q|/\rho$ and $p(\rho) = c^2\rho$.
Balance laws across junctions yield

$$\rho^e(v) = \rho^{e'}(v), \qquad \text{for all } e, e' \in \mathcal{E}(v),\ v \in \mathcal{V}_0, \quad (3)$$
$$\sum_{e\in\mathcal{E}(v)} n^e(v) q^e(v) = 0, \qquad \text{for all } v \in \mathcal{V}_0. \quad (4)$$

Input/boundary condition described by pressure at the ports

$$\rho^e(v) = u_v, \qquad \text{for all } v \in \mathcal{V}_\partial. \quad (5)$$

Example network:



- ▷ $\mathcal{V}_0 = \{v_2\}$, $\mathcal{V}_\partial = \{v_1, v_3, v_4\}$
- ▷ $\mathcal{E}(v_2) = \{e_1, e_2, e_3\}$
- ▷ $n^{e_1}(v_2) = 1$, $n^{e_2}(v_2) = -1$

Physical properties:

(P1) dissipation of energy $E = \frac{1}{2}(\|\rho\|^2 + \|q\|^2)$, i.e.

$$\frac{d}{dt}E = -(dq, q) - (q, nu)_{\mathcal{V}_\partial}$$

(P2) conservation of mass: $\frac{d}{dt}\int_{\mathcal{E}} \rho\, dt = \sum_{v\in\mathcal{V}_\partial, e\in\mathcal{E}(v)} n^e(v) q^e(v)$;

(P3) exponential convergence to equilibrium when $u \equiv 0$;

(P4) existence of unique steady states for the stationary problem.

Galerkin semidiscretization

Variational characterization: Any solution of (1)-(5) satisfies

$$(GD)\begin{cases} (a^e \partial_t \rho(t), \mu) + (\partial_x' q(t), \mu) = 0, \\ (b^e \partial_t q(t), v) - (\rho(t), \partial_x' v) + (dq(t), v) = (u(t), nv)_{\mathcal{V}_\partial}, \end{cases}$$

for all $\mu \in L^2(\mathcal{E})$ and $v \in H(div) = \{\tau : \tau^e \in H^1(e) \quad \forall e \in \mathcal{E}$ and (4) holds$\}$.

Conforming Galerkin discretization (GD_h):
Find $\rho_h \in M_h \subset L^2(\mathcal{E})$ and $q_h \in V_h \subset H(div)$, such that variational principle holds for all $\mu_h \in M_h$ and $v_h \in V_h$.

Results

Theorem. *Assume that M_h, V_h satisfy*

($A1_h$) $M_h = \partial_x' V_h$

($A2_h$) $\{r : \partial_x' r = 0\} \subset V_h$

($A3_h$) $1 \in M_h$

Then any solution of (GD_h) *fulfills (P1)-(P4).*

Further results:

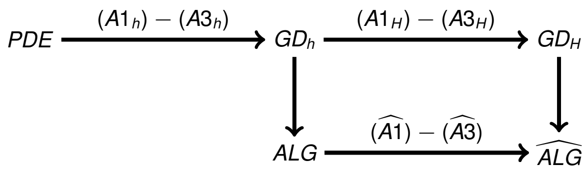
- ▷ mixed fem of arbitrary order satisfying ($A1_h$)-($A3_h$)
- ▷ stability preserving time-discretization of arbitrary order possible;
- ▷ uniform stability and error estimates for fully discrete schemes

$$\|\rho_h^n - \rho(t^n)\| + \|q_h^n - q(t^n)\| \le C(h^p + \Delta t^k) \quad \text{for all} \quad n \ge 0$$
$$\|\rho_h^n - \rho_h^\infty\| + \|q_h^n - q_h^\infty\| \le Ce^{-\alpha t^n}$$

with constants C, α independent of $n, h, \Delta t$.

Structure preserving model reduction

Model reduction by Galerkin projection can be interpreted as



Reduced model in algebraic form

$$(\widehat{ALG})\begin{cases} V_1^\top M_1 V_1 \dot z_1 + V_1^\top G V_2 z_2 = 0, \\ V_2^\top M_2 V_2 \dot z_2 - V_2^\top G^\top V_1 z_1 + V_2^\top D V_2 z_2 = V_2^\top B_2 u, \end{cases}$$

Theorem (Model reduction). *Assume that the coarse spaces M_H, V_H respectively projection matrices V_1, V_2 satisfy*

($A1_H$) $M_H = \partial_x' V_H$	$(\widehat{A1})$ $\mathcal{R}(M_1 V_1) = \mathcal{R}(GV_2)$
($A2_H$) $\{v : \partial_x' v = 0\} \subset V_H$	$(\widehat{A2})$ $\mathcal{N}(G) \subset \mathcal{R}(V_2)$
($A3_H$) $1 \in M_H$	$(\widehat{A3})$ $o_1 \in \mathcal{R}(V_2)$

Then any solution of the Galerkin approximation (GD_H) *and its algebraic representation $\widehat{ALG}$ satisfies (P1)–(P4).*

Construction of projection matrices V_1, V_2:

1. Create subspaces $\mathbb{W}_1$, $\mathbb{W}_2$ via Krylov iteration.
2. Choose finite dimensional spaces $\mathbb{Z}_1$, $\mathbb{Z}_2$, such that

$$\mathbb{V}_1 = \mathbb{W}_1 + \mathbb{Z}_1 \quad \text{and} \quad \mathbb{V}_2 = \mathbb{W}_2 + \mathbb{Z}_2$$

satisfy the compatibility conditions $(\widehat{A1}) - (\widehat{A3})$.

References:

- ▷ Egger, Kugler: *Damped wave systems on networks: Exponential stability and uniform approximations.* arXive:1605.03066.
- ▷ Egger, Kugler, Liljegren-Sailer, Marheineke: *Model reduction for wave propagation on networks.* In preparation.

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