Subproject C02

Perturbation Analysis of Hyperbolic PDAEs Describing Gas Networks

TRR 154

Mathematical modeling, simulation, and optimization using the example of gas networks

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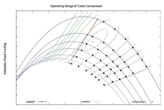
PDAE Model

Gas transport through a network $\mathcal{G} = (V, \mathcal{E})$ can be described by a set of hyperbolic PDEs

$$\begin{aligned} \partial_t p_e + \alpha_e \partial_x q_e &= 0 \\ \partial_t q_e + \beta_e \partial_x p_e &= -\gamma_e \frac{q_e|q_e|}{p_e} - \sigma_e p_e \end{aligned} \qquad e \in \mathcal{E}_\mathcal{P} \qquad \text{(ISO2)}$$

where p_e and q_e are the pressure and mass flow, α_e, β_e are pipe parameter dependent, γ_e is a friction term and σ_e accounts for a possible slope. Additional elements like compressors



Compressor-characteristics.

$$\begin{aligned} H_{ad,e} &= \kappa \left[\left(p_{in,e}/p_{out,e} \right)^{\zeta} - 1 \right] \\ H_{ad,e} &= \Phi(Q_e, n_e; A_e^n) \\ \eta_{ad,e} &= \Phi(Q_e, n_e; A_e^\eta) \\ Q_e &= c^2 q_e / p_{in,e} \end{aligned} \qquad e \in \mathcal{E}_\mathcal{C} \qquad \text{(COM)}$$

with control $p_{out,e} = p_{set,e}$ or $q_e = q_{set,e}$

valves $\quad p_{in,e} = p_{out,e} \quad q_e = 0 \quad e \in \mathcal{E}_\mathcal{S} \qquad$ (VAL-ON/OFF)

resistors $\quad p_{in,e} - p_{out,e} = \xi_e \frac{q_e|q_e|}{p_{in,e}} \quad e \in \mathcal{E}_\mathcal{R} \qquad$ (RES)

are coupled to the pipes by a set of balance equations for the flows and pairwise mappings for the pressures

$$A_R\, q_{\mathcal{P},R} + A_L\, q_{\mathcal{P},L} + A_\mathcal{C}\, q_\mathcal{C} + A_\mathcal{S}\, q_\mathcal{S} + A_\mathcal{R}\, q_\mathcal{R} = q^\Gamma \qquad (1)$$

$$B_R^\top\, p_{\mathcal{P},R} + B_L^\top\, p_{\mathcal{P},L} + B_\mathcal{C}^\top\, p_\mathcal{C} + B_\mathcal{S}^\top\, p_\mathcal{S} + B_\mathcal{R}^\top\, p_\mathcal{R} = 0. \qquad (2)$$

$$p_{e,L} = p_u^\Gamma \qquad e \in \delta^-(u), \quad u \in V_+ \qquad (3)$$

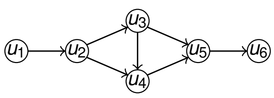
with $A = [A_R\, A_L\, A_\mathcal{C}\, A_\mathcal{S}\, A_\mathcal{R}]$ the incidence matrix of $\mathcal{G}$ and $B \in \ker |A|$.

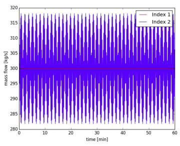
Perturbation Behaviour

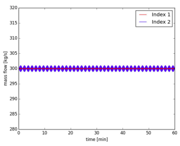
▷ Depending on the semi-discretization in space of system (ISO2) & (1)-(3), the resulting DAEs may be of arbitrary high index.

▷ Solutions may not reflect the properties of the PDAE system correctly.

Example: $V = \{u_1, \ldots, u_6\}$ $\quad V_+ = \{u_1\}$ $\quad \mathcal{E} = \{e_1, \ldots, e_7\}$







$\delta^q_{u_6} = 10^{-4} \sin(10^6 t)$ kg/s

Mass inflow at u_1, max Δx = 40 km.

Mass inflow at u_1, max Δx = 2 km.

Extension to General Networks

▷ This approach seems applicable to more general gas networks e.g., with compressors (COM)

▷ Analysis of hyperbolic PDAEs of the form

$$\begin{aligned} \mathfrak{u}' + \mathcal{B}\mathfrak{u} + \mathcal{D}(\mathfrak{u}, z, t) &= 0 \qquad \text{in } \mathcal{H} \\ g(z,t) &= 0 \end{aligned}$$

▷ g possesses some properties that have been proven useful in the treatment of elliptic and parabolic PDAEs of that form.

Perturbed Problem

We are interested in the behaviour of a solution of a pipe network described by equations (ISO2) and (1)-(3) regarding perturbations in these equations.

$$\begin{aligned} \partial_t p_e^\delta + \alpha_e \partial_x q_e^\delta &= \delta^\Omega_{e,1} \\ \partial_t q_e^\delta + \beta_e \partial_x p_e^\delta &= \delta^\Omega_{e,2} - \gamma_e \frac{q_e^\delta|q_e^\delta|}{p_e^\delta} - \sigma_e p_e^\delta \end{aligned} \qquad e \in \mathcal{E}_\mathcal{P} \qquad \text{(ISO2}'\text{)}$$

$$A_R\, q^\delta_{\mathcal{P},R} + A_L\, q^\delta_{\mathcal{P},L} = q^\Gamma + \delta^q \qquad B_R^\top p^\delta_{\mathcal{P},R} + B_L^\top p^\delta_{\mathcal{P},L} = \delta^p \qquad (1')\,(2')$$

$$p^\delta_{e,L} = p_u^\Gamma + \delta_u^p \qquad e \in \delta^-(u), \quad u \in V_+. \qquad (3')$$

Homogenization

Choosing homogenization functions for (p, q) and (p^δ, q^δ)

$$\bar{q}_e^\delta := \begin{cases} \frac{x}{\ell_e}(q_u^\Gamma + \delta_u^q) & e = e_1 \\ 0 & \text{else} \end{cases} \qquad \bar{p}_e^\delta := \begin{cases} \frac{\ell_e - x}{\ell_e}(p_u^\Gamma + \delta^p_{u,e}) & u \in V_+ \\ \frac{\ell_e - x}{\ell_e}\delta^p_{u,e} & u \notin V_+, e \in \mathcal{T} \\ \frac{\ell_e - x}{\ell_e}\delta^p_{u,e} + \frac{x}{\ell_e}\delta^p_{v,e} & \text{else} \end{cases}$$

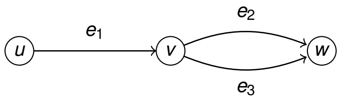
for $e = (u,v) \in \mathcal{E}$, $\delta^+(u) = \{e_1, \ldots, e_{n_u}\}$.

If $(\hat{p}^\delta, \hat{q}^\delta)$ solves (ISO2′) with (1′) - (3′), then $(p^\delta, q^\delta) = (\hat{p}^\delta - \bar{p}^\delta, \hat{q}^\delta - \bar{q}^\delta)$ solves

$$\begin{aligned} \partial_t p_e^\delta + \alpha_e \partial_x q_e^\delta &= \delta^\Omega_{e,1} - \partial_t \bar{p}^\delta - \alpha_e \partial_x \bar{q}^\delta \\ \partial_t q_e^\delta + \beta_e \partial_x p_e^\delta &= \delta^\Omega_{e,2} - f_e(p_e^\delta + \bar{p}_e^\delta, q_e^\delta + \bar{q}_e^\delta) - \partial_t \bar{q}^\delta - \beta_e \partial_x \bar{p}^\delta \end{aligned} \qquad \text{(ISO2}''\text{)}$$

for $e \in \mathcal{E}$ and (1′) - (3′) with zero right hand side. The non-linear function f_e is given by the right hand side from the 2nd equation of (ISO2).

Example:



$V_+ = \{u\}$ $\quad \mathcal{T} = \{e_1, e_2\}$

$\delta^+(v) = \{e_1\}$ $\quad \delta^+(w) = \{e_2, e_3\}$

$$\bar{q}^\delta_{e_1} = \frac{x}{\ell_1}(q_v^\Gamma + \delta_v^q) \qquad \bar{q}^\delta_{e_2} = \frac{x}{\ell_2}(q_w^\Gamma + \delta_w^q) \qquad \bar{q}^\delta_{e_3} = 0$$

$$\bar{p}^\delta_{e_1} = \frac{\ell_1 - x}{\ell_1}(p_u^\Gamma + \delta^p_{u1}) \qquad \bar{p}^\delta_{e_2} = \frac{\ell_2 - x}{\ell_2}\delta^p_{w2} \qquad \bar{p}^\delta_{e_3} = \frac{\ell_3 - x}{\ell_3}\delta^p_{w3} + \frac{x}{\ell_3}\delta^p_{v3}$$

Perturbation Analysis

After homogenization of both systems, the perturbation analysis reduces to the analysis of

$$\mathfrak{u}' - \mathfrak{u}^{\delta\prime} + \mathcal{B}(\mathfrak{u} - \mathfrak{u}^\delta) + \mathcal{D}(\mathfrak{u}, \mathfrak{u}^\delta, t) = \mathcal{F}(\delta^\Omega, \delta^\Gamma, \delta^{\Gamma\prime}) \qquad \text{in } \mathcal{H}$$

in an appropriate function space $\mathcal{H}$. Here $\mathfrak{u} = (p, q)$, $\mathfrak{u}^\delta = (p^\delta, q^\delta)$.

Theorem 1 (A priori estimates). *Let the boundary data p^Γ, q^Γ and the perturbations δ^Ω and δ^Γ and their first derivatives w.r.t. time be bounded. And let the velocity of each pipe $e \in \mathcal{E}$ be bounded by $|\nu_e| \le \bar{\nu}$. Then (p, q) and (p^δ, q^δ) and their first derivatives w.r.t. time are bounded as well.*

Theorem 2 (Perturbation result). *If the assumptions of Theorem 1 hold, we can derive that*

$$\max_{\mathcal{T}} \|\mathfrak{u} - \mathfrak{u}^\delta\|^2_{L^2} \le K \left(\|\delta_0\|^2_{L^2} + \max_{\mathcal{T}} \|\delta^\Omega\|^2_{L^2} + \max_{\mathcal{T}} |\delta^\Gamma| \right)$$

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