Subproject B04

TRR 154

Mathematical modeling, simulation, and optimization using the example of gas networks

Nonlinear probabilistic constraints in gas transportation problems

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Topic and Challenges

The aim of this project consists in applying nonlinear probabilistic constraints to optimization problems in gas transportation assuming the underlying random parameter obeys a multivariate and continuous distribution. Robustness in the sense of probabilistic network design shall be facilitated.

- ▷ Modeling of uncertain parameters as random vectors with multivariate distribution, taking network node correlations into consideration
- ▷ Using continuous distributions for higher efficiency compared with discretization of the random vector
- ▷ Optimization in gas transport networks requires non-linear and even implicit probabilistic constrains
- ▷ There are no analytical representations for both function values and gradients of the probability function

Probabilistic Constraints

Stochastic optimization problem with probabilistic constraints:

$$\min\{f(x) \mid \varphi(x) \geq p,\ x \in X\}$$

Probability function: $\varphi(x) := \mathbb{P}\big(g(x,\xi) \leq 0\big)$

In general, both values and gradients are needed for $\varphi(\cdot)$:

Spheric-Radial Decomposition: $\xi \in \mathcal{N}(0,\Sigma)$ with $\Sigma = LL^\top$

If $g(x,\cdot)$ continuous, convex and x such that $g(x,0) \leq 0$. Then

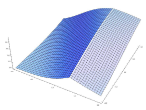
$$\varphi(x) = \int_{v\in\mathbb{S}^{n-1}} \chi_{\text{cdf}}(\rho(x,v))\,d\mu_\eta(v)\,,$$

where $\rho(x,v) := \sup\{r \geq 0 \mid g(x, rLv) \leq 0\}$.

Additionally, if $g : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}$ continuously differentiable, then

$$\nabla\varphi(x) = \int_{v\in\mathbb{S}^{n-1}} -\frac{\chi_{\text{pdf}}(\rho(x,v))}{\langle \nabla_\xi g(x,\rho(x,v)Lv), Lv\rangle} \nabla_x g(x,\rho(x,v)Lv)\,d\mu_\eta(v)$$

In general: Smooth g, ξ do not imply smooth probability functions

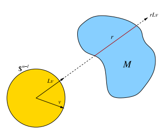


$\Longrightarrow$ Derivatives (subdifferential) in terms of Clarke or Mordukhovich

Spheric-Radial Decomposition

Let be $\xi \sim \mathcal{N}(0,\Sigma)$ n-dimensional Gaussian random vector with covariance $\Sigma = LL^\top$. Then it holds:

$$\mathbb{P}(\xi \in M) = \int_{\mathbb{S}^{n-1}} \chi\{r \geq 0 \mid rLv \in M\}\,d\mu_\eta(v),$$



where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, μ_η the law of uniform distribution on it. χ is the law of chi-distribution with n degree of freedom.

Stationary Gas Networks

Feasibility of exit load nomination b for pressure bounds $p^{min/max}$

$$\begin{aligned}
\exists z:\quad A_N^\top g(b,z) &= \Phi_N|z|z\\
\min_{k=1,\dots,|\mathcal{V}|}\left[(p_k^{max})^2 + g_k(b,z)\right] &\geq \max_{k=1,\dots,|\mathcal{V}|}\left[(p_k^{min})^2 + g_k(b,z)\right]\\
(p_0^{min})^2 &\leq \min_{k=1,\dots,|\mathcal{V}|}\left[(p_k^{max})^2 + g_k(b,z)\right]\\
(p_0^{max})^2 &\geq \max_{k=1,\dots,|\mathcal{V}|}\left[(p_k^{min})^2 + g_k(b,z)\right]
\end{aligned}$$

Definition: $g(u,v) := \left(A_B^\top\right)^{-1}\Phi_B|A_B^{-1}(u - A_N v)|A_B^{-1}(u - A_N v)$
$A = (A_B|A_N)$ incidence matrix, $\Phi = (\Phi_B|\Phi_N)$ frictional coefficients

Uncertainty: Nomination b assumed to be of Gaussian type

Joint Robust/Probabilistic Approach

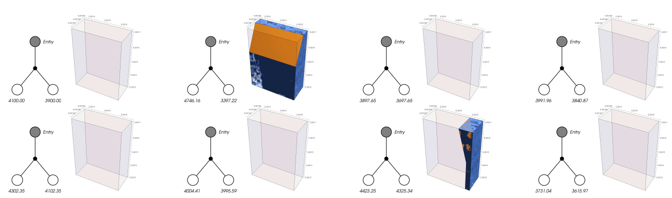
Two different characters of uncertainty:

1. Gas demand (nominations) $\Rightarrow$ Distribution available
2. Friction coefficients $\Rightarrow$ No statistical information available

Find maximum uncertainty allowed for friction coefficients while guaranteeing demand satisfaction at high probability:

$$\max\left\{f(\delta)\ \middle|\ \mathbb{P}\big(g(\Phi,b) \leq 0 \quad \forall \Phi \in U_\delta\big) \geq p\right\}$$

U_δ ... uncertainty set (e.g. rectangle, ellipsoid)
δ ... certain shape parameter (e.g. edge length, axis)



Out-of-sample constraint violation ($p = 0.80$) for fixed uncertainty set

References

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[3] H.Heitsch, H.Leövey and W.Römisch: Are Quasi-Monte Carlo algorithms efficient for two-stage stochastic programs?, COAP (2016), DOI 10.1007/s10589-016-9843-z.

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