Logic and Computability

Soundness and Completeness of Natural Deduction

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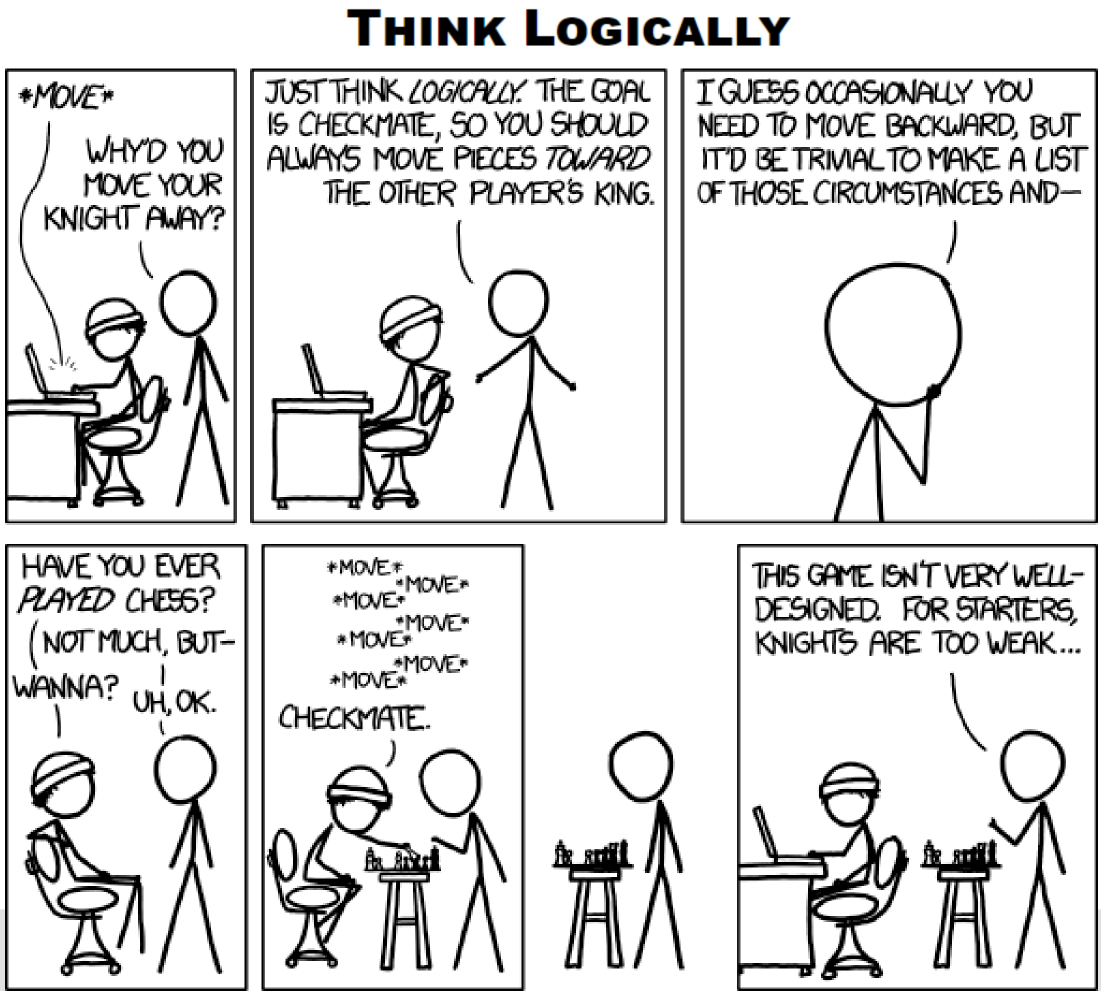


https://xkcd.com/1112/

Recap – Semantic Entailment

$$\underbrace{\phi_1,\ \phi_2,\ \ldots,\ \phi_n}_{\textbf{Premises}} \vDash \underbrace{\Psi}_{\textbf{Conclusion}}$$

Any model that is a **satisfying model** for $\boldsymbol{\phi_1, \ldots, \phi_n}$, is also a **satisfying model** for $\boldsymbol{\psi}$

Recap – Syntactic Entailment

$$\underbrace{\phi_1,\ \phi_2,\ \dots,\ \phi_n}_{\text{Premises}} \vdash \underbrace{\Psi}_{\text{Conclusion}}$$

From $\phi_1 \dots \phi_n$ we can (syntactically) **prove** that ψ holds (via Natural Deduction)

Recap – Soundness of ND for Prop. Logic

- Definition

$$\boldsymbol{\phi_1, \phi_2, \dots, \phi_n \vdash \psi} \quad \Rightarrow \quad \boldsymbol{\phi_1, \phi_2, \dots, \phi_n \vDash \psi}$$

Correct **syntactic** entailment
From $\phi_1 \dots \phi_n$ we can **prove** that ψ holds.

Correct **semantic** entailment
Each model that satisfies all premises $\phi_1 \dots \phi_n$ also satisfies ψ.

Recap – Soundness of ND for Prop. Logic

- Definition

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi \qquad \Rightarrow \qquad \phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

- Consequence of Soundness
  - $\phi_1, \phi_2, \ldots, \phi_n \nvDash \psi \qquad \Rightarrow \qquad \phi_1, \phi_2, \ldots, \phi_n \nvdash \psi$
  - Thus, a single **counterexample** is sufficient to show that sequent is not provable.
  - $\mathcal{M}$ is a counterexample if

    $\mathcal{M}$ satisfies all premises, and $\mathcal{M}$ does not satisfy the conclusion.

Recap – Completeness of ND for Prop. Logic

- Definition

$$\boldsymbol{\phi_1, \phi_2, \ldots, \phi_n \vDash \psi} \quad \Rightarrow \quad \boldsymbol{\phi_1, \phi_2, \ldots, \phi_n \vdash \psi}$$

Correct **semantic** entailment

Each model that satisfies all premises $\phi_1 \ldots \phi_n$ also satisfies ψ.

Correct **syntactic** entailment

From $\phi_1 \ldots \phi_n$ we can **prove** that ψ holds.

Recap – Completeness of ND for Prop. Logic

- Definition

$$\boldsymbol{\phi_1, \phi_2, \dots, \phi_n \vDash \psi} \qquad \Rightarrow \qquad \boldsymbol{\phi_1, \phi_2, \dots, \phi_n \vdash \psi}$$

- Consequences of Completeness
  - Unprovable sequents are incorrect entailments.
    - $\phi_1, \phi_2, \dots, \phi_n \nvdash \psi \qquad \Rightarrow \qquad \phi_1, \phi_2, \dots, \phi_n \nvDash \psi$

Where is the proof?

Theorem

- Natural Deduction for Propositional Logic is **sound**:
  - $\phi_1, \phi_2, \dots, \phi_n \vdash \psi \quad \Rightarrow \quad \phi_1, \phi_2, \dots, \phi_n \vDash \psi$

Proof ? 

- Natural Deduction for Propositional Logic is **complete**:
  - $\phi_1, \phi_2, \dots, \phi_n \vDash \psi \quad \Rightarrow \quad \phi_1, \phi_2, \dots, \phi_n \vdash \psi$

Proof ? 

Learning Outcomes

After this lecture…

1. students can explain the concepts of soundness and completeness of natural deduction for propositional logic.

2. students can sketch the proof for soundness and completeness of natural deduction for propositional logic.

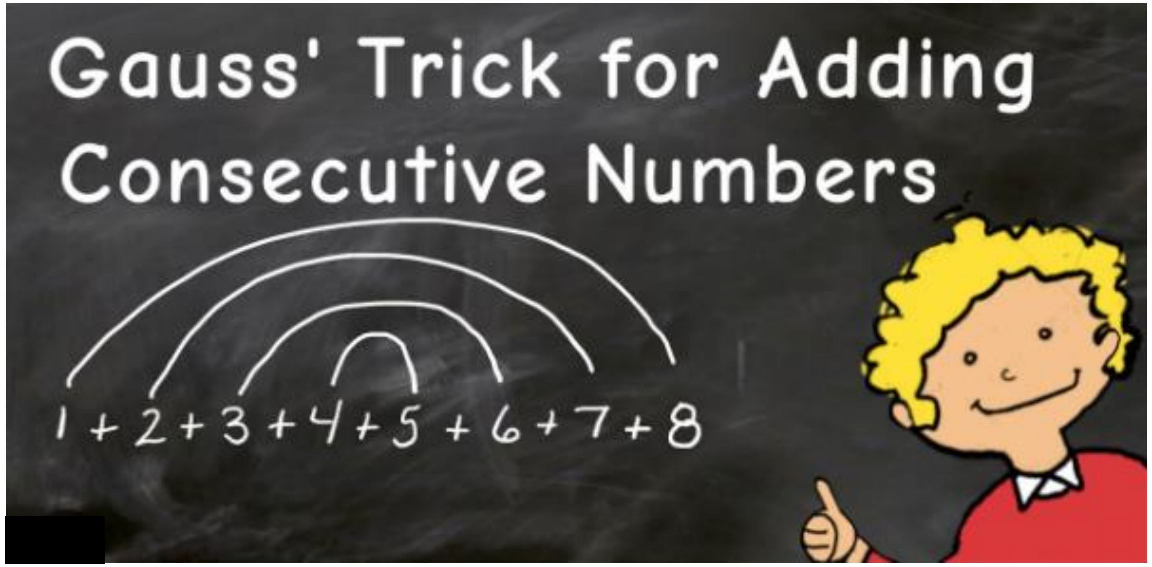
3. students can perform a deduction proof for tautologies based on the structure of the completeness proof.

Outline

- Recap: Mathematical Induction
- Prove that ND for prop. logic is **sound**
- Prove that ND for prop. logic is **complete**
- Prove tautologies with uniform method
  - from completeness proof

Mathematical Induction

- Induction can prove equations for arbitrary n
- Example
  - $1 + 2 + 3 + 4 + \cdots + n = \frac{n \cdot (n+1)}{2}$



Principle of Induction

- Show that *every natural number* satisfies a certain property M
  - We write $M(5)$ for the property is true for 5
  - We write $M(101)$ for the property is true for 101
  - …

Induction Hypothesis

- For every $n \in \mathbb{N}$, the property $M(n)$ holds.

Base Case

- The number 1 has property M, i.e., we have a proof of $M(1)$.

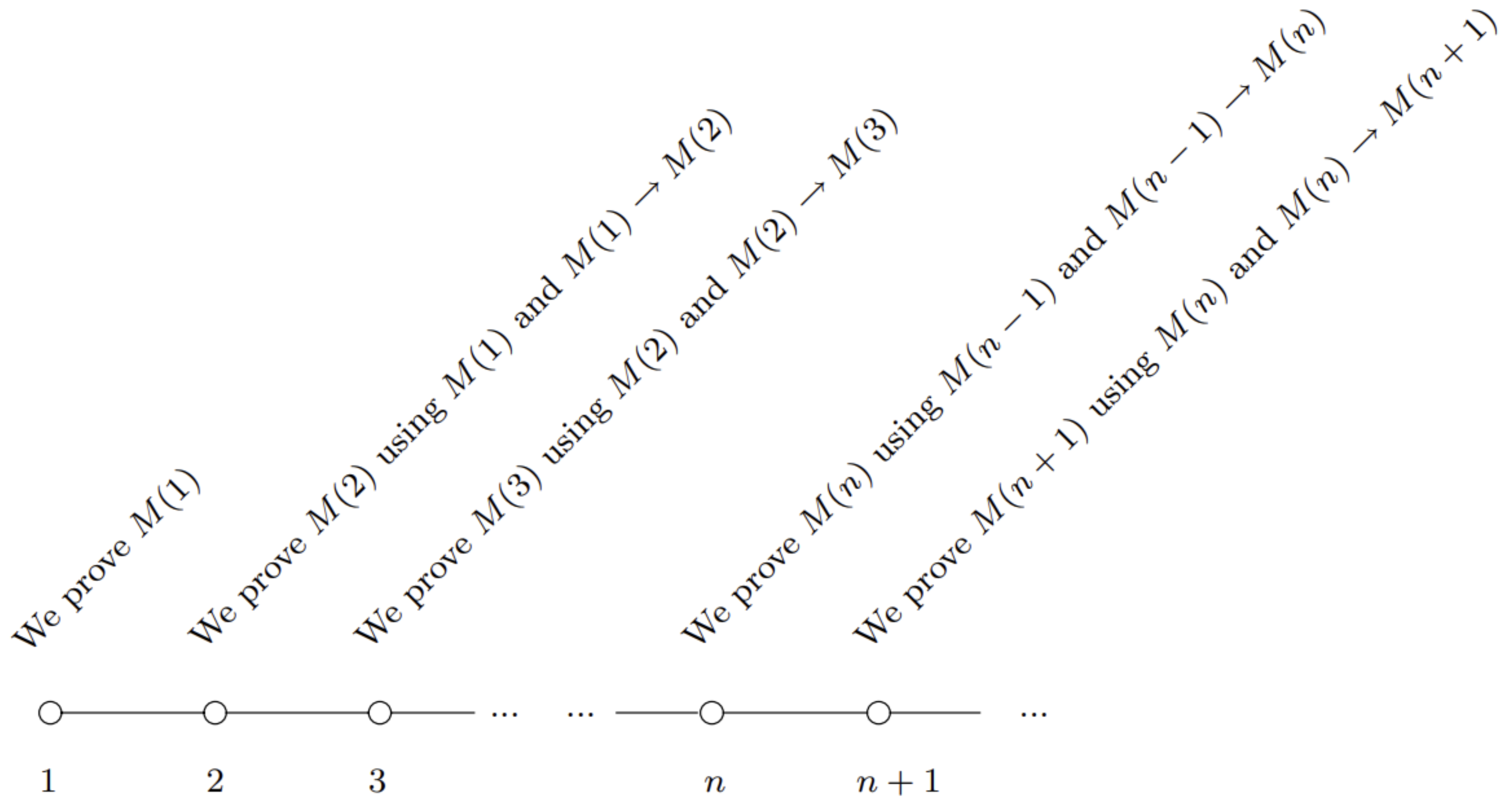
Induction Step

- If we assume that $\mathrm{M}(n)$ holds, we can show $M(n+1)$ holds as well, i.e., we have a proof of $M(n) \rightarrow M(n+1)$

Proves Induction Hypothesis

Principle of Induction

- By proving just two facts, $\boldsymbol{M(1)}$ and $\boldsymbol{M(n) \rightarrow M(n+1)}$ for a unconstrained number n, we are able to deduce $\boldsymbol{M(k)}$ for each natural number k.



Mathematical Induction- Gauss' Example

Induction Hypthesis

- We assume $\mathrm{LHS}_n = \mathrm{RHS}_n$

Base Case

- $\mathrm{LHS}_1 = 1$
- $\mathrm{RHS}_1 = \frac{1\cdot(1+1)}{2} = 1$

Notation:

- LHS_n for $1+2+3+4+\cdots+n$
- RHS_n for $\frac{n\cdot(n+1)}{2}$

Mathematical Induction- Gauss’ Example

Notation:

- LHS_n for $1 + 2 + 3 + 4 + \cdots + n$
- RHS_n for $\frac{n\cdot(n+1)}{2}$

Induction Hypthesis

- We assume $\mathrm{LHS}_n = \mathrm{RHS}_n$

Induction Step

$\boldsymbol{LHS_{n+1}}$ = $1 + 2 + 3 + .. + (n + 1)$

- $= \mathrm{LHS}_n + (\mathrm{n} + 1)$
- $= \mathrm{RHS}_n + (\mathrm{n} + 1)$ (by our induction hypothesis)
- $= \frac{n\cdot(n+1)}{2} + (n + 1)$
- $= \frac{n\cdot(n+1)}{2} + \frac{2\cdot(n+1)}{2}$
- $= \frac{(n+2)\cdot(n+1)}{2}$
- $= \frac{((n+1)+1)\cdot(n+1)}{2} = \boldsymbol{RHS_{n+1}}$

Course-of-Values Induction

- Variant of mathematical induction

Induction Hypothesis:

- $T(1) \wedge T(2) \wedge \cdots \wedge T(n)$ holds.

Base Case:

- $T(1)$.

Induction Step:

- Prove that $T(1) \wedge T(2) \wedge \cdots \wedge T(n) \rightarrow T(n+1)$

Proves Induction Hypothesis

Outline

- Recap: Mathematical Induction
- Prove that ND for prop. logic is **sound**
- Prove that ND for prop. logic is **complete**
- Prove tautologies with uniform method
  - from completeness proof

Proof for Soundness for ND for Prop Logic

Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional logic formulas.

Theorem Soundness

- „If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.”

Proof for Soundness for ND for Prop Logic

Proof Idea - mathematical induction on the length of the Natural Deduction proof.

- We define the assertion $\mathrm{M}(k)$:

„For all sequents $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ which have a **proof of length k**,
it is the case that $\phi_1, \phi_2, \dots, \phi_n \vDash \psi$ holds."

- We indent to show the assertion $\mathrm{M}(k)$ by **course-of-values induction** on k

Proof for Soundness for ND for Prop Logic

Induction Hypothesis:

- $M(1) \land M(2) \land \cdots \land M(k-1)$ holds, with
  $\mathrm{M}(i)$: „For all sequents $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ which have a **proof of length i**,
  it is the case that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds."

Base Case: $\mathbf{M(1)}$ holds

- If the proof has length 1 $(k = 1)$ then it must be of the form

1	ϕ	premise

- Thus, the squent must be of the form $\phi \vdash \phi$.
  Does $\phi \vdash \phi$ imply $\phi \vDash \phi$?

Proof for Soundness for ND for Prop Logic

Induction Hypothesis:

- $M(1) \wedge M(2) \wedge \cdots \wedge M(k-1)$ holds, with
  $\mathrm{M}(i)$: „For all sequents $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ which have a **proof of length i**,
  it is the case that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds."

Base Case: $\mathbf{M(1)}$ holds

- If the proof has length 1 $(k = 1)$ then it must be of the form

1	ϕ	premise

Thus, the squent must be of the form $\phi \vdash \phi$.
YES: if ϕ evaluates to **T** so does ϕ. Thus, $\phi \vDash \phi$ holds as claimed.

Proof for Soundness for ND for Prop Logic

$\mathrm{M}(i)$: „For all sequents $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ which have a **proof of length i**,
it is the case that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds."

Inductive step: $\mathbf{M(1) \land M(2) \land \cdots \land M(k-1) \rightarrow M(k)}$

- We do not know the **last rule** that was applied!
- → Consider each rule in turn
  - $\land \boldsymbol{i}$
  - $\neg \boldsymbol{e}$
  - $\lor \boldsymbol{e}$
  - …

Structure of ND Proof

1	ϕ_1	premise
2	ϕ_2	premise
	$\vdots$	
n	ϕ_n	premise
	$\vdots$	
k	ψ	justification

Proof for Soundness for ND for Prop Logic

Inductive step: $M(1) \wedge \cdots \wedge M(k-1) \rightarrow M(k)$ with $\wedge i$ as last rule

- We have a proof $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1$ with length $< k$
- We have a proof $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_2$ with length $< k$

- Using the induction hypothesis, we conclude
  - $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi_1$
  - $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi_2$

- These two relations imply $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi_1 \wedge \psi_2$
  - **WHY?**

1	ϕ_1	premise
2	ϕ_2	premise
	$\vdots$	
k_1	ψ_1	
	$\vdots$	
k_2	ψ_2	
	$\vdots$	
k	$\psi_1 \wedge \psi_2$	$\wedge i, k_1 k_2$

Proof for Soundness for ND for Prop Logic

Why does $\boldsymbol{\phi} \vDash \boldsymbol{\psi_1}$ and $\boldsymbol{\phi} \vDash \boldsymbol{\psi_2}$ imply $\boldsymbol{\phi} \vDash \boldsymbol{\psi_1} \land \boldsymbol{\psi_2}$?

Proof for Soundness for ND for Prop Logic

- Why does $\boldsymbol{\phi} \vDash \boldsymbol{\psi_1}$ and $\boldsymbol{\phi} \vDash \boldsymbol{\psi_2}$ imply $\boldsymbol{\phi} \vDash \boldsymbol{\psi_1} \land \boldsymbol{\psi_2}$?
  - Show that the following formula is valid:
    $\eta = ((\phi \to \psi_1) \land (\phi \to \psi_2)) \to (\phi \to (\psi_1 \land \psi_2))$

$\boldsymbol{\phi}$	$\boldsymbol{\psi_1}$	$\boldsymbol{\psi_2}$	$\phi \to \psi_1$	$\phi \to \psi_2$	$((\phi \to \psi_1) \land (\phi \to \psi_2))$	$\phi \to (\psi_1 \land \psi_2)$	η
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	1	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	1	1
1	1	1	1	1	1	1	1

Proof for Soundness for ND for Prop Logic

Inductive step: $\mathbf{M(1)} \land \boldsymbol{M(2)} \land \cdots \land \boldsymbol{M(k-1)} \to \boldsymbol{M(k)}$

- → Consider each possible last rule
  - $\land \boldsymbol{i}$ ✓
    - **Inductive step done:**
      → For all proofs of length k with $\land$ ***i* as last rule** it holds that:
      „ If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$, then $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$"
  - $\neg \boldsymbol{e}$ ⇐ **Next**
  - $\lor \boldsymbol{e}$
  - …

Proof for Soundness for ND for Prop Logic

Inductive step: $M(1) \wedge \cdots \wedge M(k-1) \rightarrow M(k)$ with $\neg e$ as last rule

- We have a proof $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ with length $k_1 < k$
- We have a proof $\phi_1, \phi_2, \ldots, \phi_n \vdash \neg\psi$ with length $k_2 < k$

- Using the induction hypothesis, we conclude
  - $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$
  - $\phi_1, \phi_2, \ldots, \phi_n \vDash \neg\psi$

- These two relations imply $\phi_1, \phi_2, \ldots, \phi_n \vDash \bot$
  - **WHY?**

1	ϕ_1	premise
2	ϕ_2	premise
	$\vdots$	
k_1	ψ	
	$\vdots$	
k_2	$\neg\psi$	
	$\vdots$	
k	$\bot$	$\neg e, k_1, k_2$

Proof for Soundness for ND for Prop Logic

Why does $\boldsymbol{\phi} \vDash \boldsymbol{\psi}$ and $\boldsymbol{\phi} \vDash \neg\boldsymbol{\psi}$ imply $\boldsymbol{\phi} \vDash \bot$?

Proof for Soundness for ND for Prop Logic

- Why does $\boldsymbol{\phi} \vDash \boldsymbol{\psi}$ and $\boldsymbol{\phi} \vDash \neg\boldsymbol{\psi}$ imply $\boldsymbol{\phi} \vDash \bot$?

Show that the following formula is valid:

$$\eta = ((\phi \to \psi) \land (\phi \to \neg\psi)) \to (\phi \to \bot)$$

$\boldsymbol{\phi}$	$\boldsymbol{\psi}$	$\phi \to \boldsymbol{\psi}$	$\phi \to \neg\boldsymbol{\psi}$	$(\phi \to \psi) \land (\phi \to \neg\psi)$	$\phi \to \bot$	η
0	0	1	1	1	1	1
0	1	1	1	1	1	1
1	0	0	1	0	0	1
1	1	1	0	0	0	1

Proof for Soundness for ND for Prop Logic

Inductive step: $\mathbf{M(1)} \wedge \boldsymbol{M(2)} \wedge \cdots \wedge \boldsymbol{M(k-1)} \rightarrow \boldsymbol{M(k)}$

- → Consider each possible last rule
  - $\wedge \boldsymbol{i}$ ✓
  - $\neg \boldsymbol{e}$ ✓
    - **Inductive step done:**
      → For all proofs of length k with $\wedge$ $\boldsymbol{i}$ **or** $\neg\boldsymbol{e}$ **as last rule** it holds that: „ If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$, then $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$"
  - $\vee$ $\boldsymbol{e}$ ⇐ **Next**
  - $\perp$ $\boldsymbol{e}$ ⇐ **Next**

- **Try the induction step for $\vee$ $\boldsymbol{e}$ or $\perp$ $\boldsymbol{e}$ as last rule**

Proof for Soundness for ND for Prop Logic

Inductive step: $M(1) \wedge \cdots \wedge M(k-1) \rightarrow M(k)$ with $\bot\ e$ as last rule

- We have **proofs** with **length $< k$ for:**
  - $\phi_1, \phi_2, \ldots, \phi_n \vdash \bot$
- Using the **induction hypothesis**, we conclude
  - $\phi_1, \phi_2, \ldots, \phi_n \vDash \bot$
- This relation implies $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$
  - **Show with truth table as before**

1	ϕ_1	premise
2	ϕ_2	premise
	$\vdots$	
n	ϕ_n	premise
	$\vdots$	
k_1	$\bot$	
	$\vdots$	
k	ψ	$\bot e\ k_1$

Proof for Soundness for ND for Prop Logic

- Why does $\boldsymbol{\phi} \vDash \bot$ imply $\boldsymbol{\phi} \vDash \boldsymbol{\psi}$?

Show that the following formula is valid:

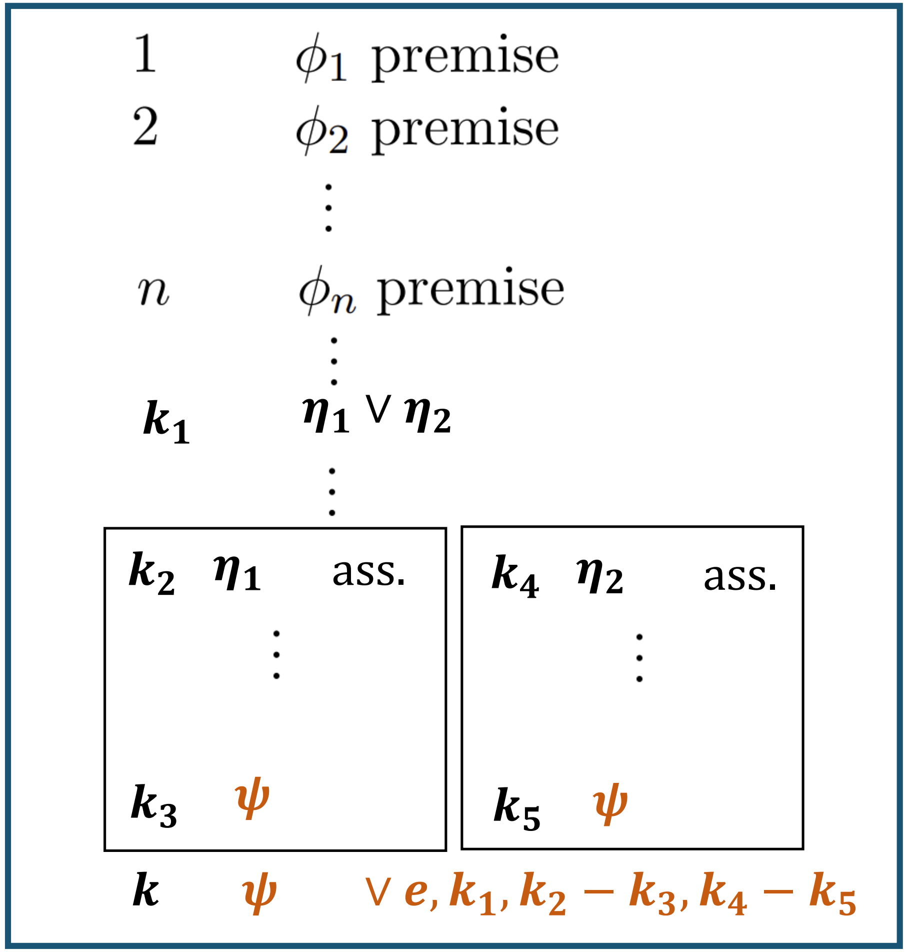
$\eta = (\phi \rightarrow \bot) \rightarrow (\phi \rightarrow \psi)$

$\boldsymbol{\phi}$	$\boldsymbol{\psi}$	$\phi \rightarrow \bot$	$\phi \rightarrow \boldsymbol{\psi}$	η
0	0	1	1	1
0	1	1	1	1
1	0	0	0	1
1	1	0	1	1

Proof for Soundness for ND for Prop Logic

Inductive step: $M(1) \wedge \cdots \wedge M(k-1) \rightarrow M(k)$ with $\vee e$ as last rule

- We have **proofs** with **length $< k$ for:**
  - $\phi_1, \phi_2, \ldots, \phi_n \vdash \eta_1 \vee \eta_2$
  - $\phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vdash \psi$
  - $\phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vdash \psi$
- Using the **induction hypothesis**, we conclude
  - $\phi_1, \phi_2, \ldots, \phi_n \vDash \eta_1 \vee \eta_2$
  - $\phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vDash \psi$
  - $\phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vDash \psi$
- These three relations imply $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$
  - **Show with truth table as before**



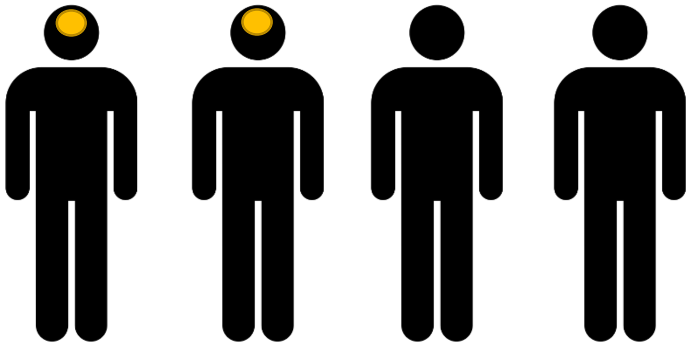
Outline

- Recap: Mathematical Induction
- Prove that ND for prop. logic is **sound**
- **10 min Coffee Break!**
- Prove that ND for prop. logic is **complete**
- Prove tautologies with uniform method
  - from completeness proof

Puzzle

1. There are n prisoners.
2. Each prisoner has either a mark (e.g., a dot) on their head or not. Each prisoner can see the marks on the other prisoners but not their own.
3. If a prisoner is sure that they have a mark, they must leave the prison the next day.
4. Every prisoner knows that every other prisoner has the same information and acts rationally.
5. There is at least one prisoner with a mark.

When will the last prisoner with a marking leave the prison?



Puzzle

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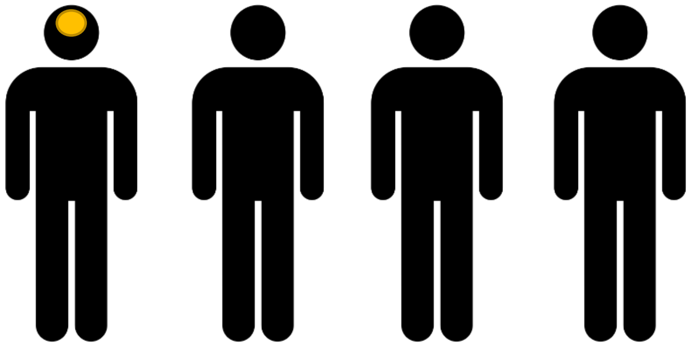
3. If a prisoner is sure that they have a mark, they must leave the prison the next day.
4. Every prisoner knows that every other prisoner has the same information and acts rationally.
5. There is at least one prisoner with a mark.

When will the last prisoner with a marking leave the prison?

k: number of people with a mark

k=1 → Everyone with a marking leaves on day 1

- **WHY**? A prisoner sees no marked heads and thus immediately knows that they must be the only one with a mark.
  They leave the prison the next day.



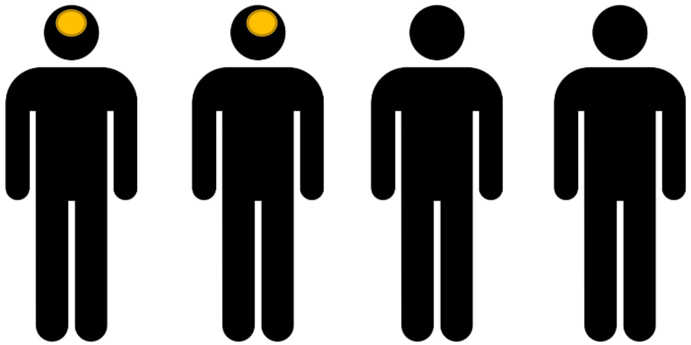
Puzzle

1. There are n prisoners.
2. Each prisoner has either a mark (e.g., a dot) on their head or not. Each prisoner can see the marks on the other prisoners but not their own.
3. If a prisoner is sure that they have a mark, they must leave the prison the next day.
4. Every prisoner knows that every other prisoner has the same information and acts rationally.
5. There is at least one prisoner with a mark.

When will the last prisoner with a marking leave the prison?

k: number of people with a mark

k=2 → Everyone with a marking leaves on day 2



- **WHY**? Each of the two prisoners with marking sees one marked head. Each waits one day to see if the other prisoner leaves the prison. Since the other prisoner does not leave on the first day, each realizes there must be two marked prisoners. Therefore, both leave on the second day.

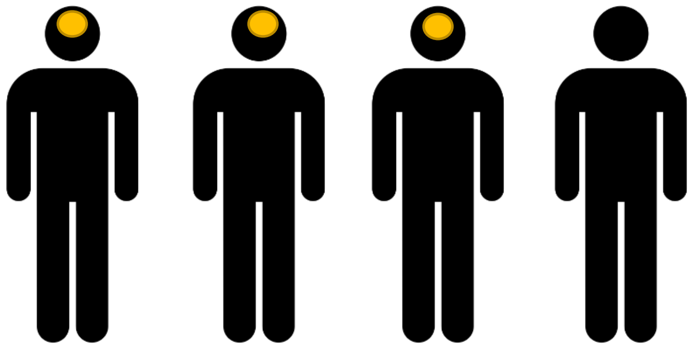
Puzzle

1. There are n prisoners.
2. Each prisoner has either a mark (e.g., a dot) on their head or not. Each prisoner can see the marks on the other prisoners but not their own.
3. If a prisoner is sure that they have a mark, they must leave the prison the next day.
4. Every prisoner knows that every other prisoner has the same information and acts rationally.
5. There is at least one prisoner with a mark.

When will the last prisoner with a marking leave the prison?

k: number of people with a mark

$k = 3$ → Everyone with a marking leaves on day 3



- Each marked prisoner sees 2 marked heads.
- Each waits 2 days to see if those 2 prisoners leave the prison.
- Since those 2 prisoners do not leave, each marked prisoner realizes there must be exactly 3 marked prisoners.
- All marked prisoners leave on the 3-th day.

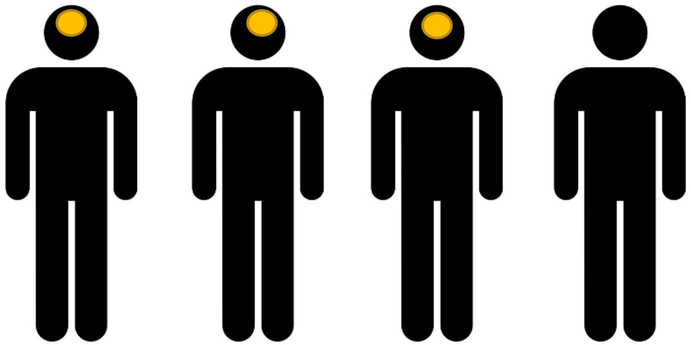
Puzzle

1. There are n prisoners.
2. Each prisoner has either a mark (e.g., a dot) on their head or not. Each prisoner can see the marks on the other prisoners but not their own.
3. If a prisoner is sure that they have a mark, they must leave the prison the next day.
4. Every prisoner knows that every other prisoner has the same information and acts rationally.
5. There is at least one prisoner with a mark.

When will the last prisoner with a marking leave the prison?

k: number of people with a mark

General case k → Everyone with a marking leaves on day k



- Each marked prisoner sees k−1 marked heads.
- Each waits k−1 days to see if those k−1 prisoners leave the prison.
- Since those k−1 prisoners do not leave, each marked prisoner realizes there must be exactly k marked prisoners.
- All marked prisoners leave on the k-th day.

Outline

- Recap: Mathematical Induction
- Prove that ND for prop. logic is **sound**
- **10 min Coffee Break!**
  - **Afterwards: Warm-up Puzzle**
- **Prove that ND for prop. logic is complete**
- Prove tautologies with uniform method
  - from completeness proof

Proof for Completeness of ND for Prop Logic

Let $\phi_1, \phi_2, \dots, \phi_n$ and ψ be propositional logic formulas.

Theorem Completeness

- „If $\phi_1, \phi_2, \dots, \phi_n \vDash \psi$ is valid, then $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ holds."

Proof for Completeness of ND for Prop Logic

Proof Idea:

- Assuming that $\phi_1, \phi_2, \dots, \phi_n \vDash \psi$ holds:
- Step 1: We show that $\vDash \phi_1 \rightarrow \left(\phi_2 \rightarrow (\dots(\phi_n \rightarrow \psi)\dots)\right)$ holds.
- Step 2: We show that $\vdash \phi_1 \rightarrow \left(\phi_2 \rightarrow (\dots(\phi_n \rightarrow \psi)\dots)\right)$ holds.
- Step 3: Finally, we show that $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ holds.

Proof for Completeness of ND for Prop Logic

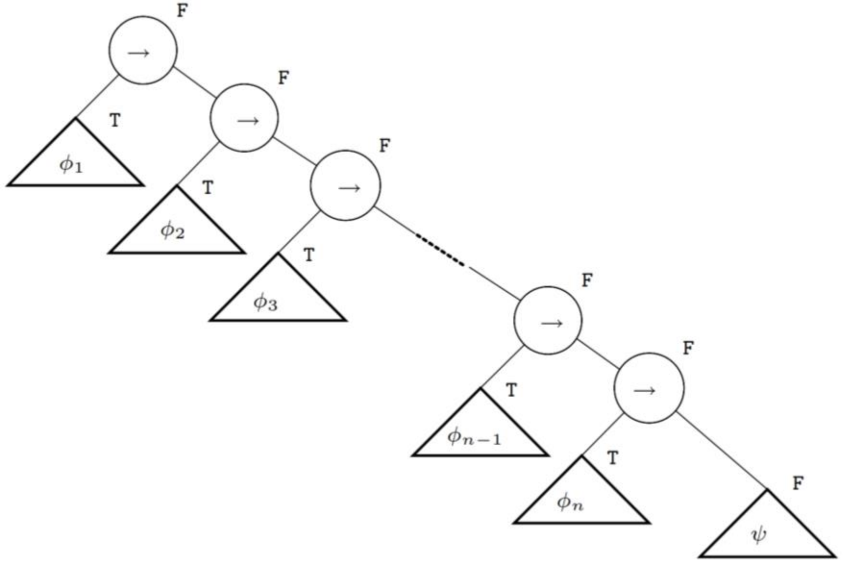
Proof for Step 1: $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ **implies** $\vDash \phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))$

- Note: $\vDash \phi$ means that ϕ is **valid**
  - ϕ evaluates to true under any model.
  - e.g., $\vDash (a \vee \neg a)$

Proof for Completeness of ND for Prop Logic

Proof for Step 1: From $\boldsymbol{\phi_1, \phi_2, \dots, \phi_n \vDash \psi}$, prove $\boldsymbol{\vDash \phi_1 \to (\phi_2 \to (\dots(\phi_n \to \psi) \dots))}$.

- $\boldsymbol{\phi_1 \to (\phi_2 \to (\dots(\phi_n \to \psi) \dots))}$ could only evaluate to false
  - if all $\phi_1 \dots \phi_n$ evaluate to true, but ψ evaluates to false



- But this contradicts the fact that $\phi_1, \phi_2 \dots \phi_n \vDash \psi$ holds.
- Thus, $\vDash \boldsymbol{\phi_1 \to (\phi_2 \to (\dots(\phi_n \to \psi) \dots))}$ holds.

Proof for Completeness of ND for Prop Logic

Proof Idea:

- Assuming that $\phi_1, \phi_2, \dots, \phi_n \vDash \psi$ holds:
- Step 1: We show that $\vDash \phi_1 \rightarrow \big(\phi_2 \rightarrow (\dots(\phi_n \rightarrow \psi)\dots)\big)$ holds. ✓
- Step 2: We show that $\vdash \phi_1 \rightarrow \big(\phi_2 \rightarrow (\dots(\phi_n \rightarrow \psi)\dots)\big)$ holds.
- **Step 3: Finally, we show that $\boldsymbol{\phi_1, \phi_2, \dots, \phi_n \vdash \psi}$ holds.** ⇦

Proof for Completeness of ND for Prop Logic

Proof for Step 3: From $\vdash \boldsymbol{\phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))}$ prove $\boldsymbol{\phi_1, \phi_2, \ldots, \phi_n \vdash \psi}$.

- We have a proof for

$\vdash \phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))$

$\vdots$

$\boldsymbol{\phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))}$

Proof for Completeness of ND for Prop Logic

Proof for Step 3: From $\vdash \boldsymbol{\phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))}$ **prove** $\boldsymbol{\phi_1, \phi_2, \ldots, \phi_n \vdash \psi}$.

- We have a proof for $\vdash \phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))$
- Transform into a proof for $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

$\vdots$

$\boldsymbol{\phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))}$

$\vdots$

$\boldsymbol{\phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))}$

Proof for Completeness of ND for Prop Logic

Proof for Step 3: From $\vdash \phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))$ prove $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$.

- We have a proof for $\vdash \phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))$

$$\begin{array}{l} \vdots \\ \phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots)) \end{array}$$

- Transform into a proof for $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

ϕ_1	ass.
ϕ_2	ass.
$\vdots$	
ϕ_n	ass.
$\vdots$	
$\phi_1 \to (\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))$	$\to e$
$\phi_2 \to (\ldots(\phi_n \to \psi)\ldots))$	$\to e$
$\vdots$	
$\phi_n \to \psi$	$\to e$
ψ	$\to e$

Proof for Completeness of ND for Prop Logic

Proof Idea:

- Assuming that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds:
- Step 1: We show that $\vDash \phi_1 \rightarrow \left(\phi_2 \rightarrow (\ldots(\phi_n \rightarrow \psi)\ldots)\right)$ holds.
- **Step 2: We show that $\vdash \boldsymbol{\phi_1 \rightarrow \left(\phi_2 \rightarrow (\ldots(\phi_n \rightarrow \psi)\ldots)\right)}$ holds.**
- Step 3: Finally, we show that $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ holds.

Proof for Completeness of ND for Prop Logic

Proof for Step 2:

From $\vDash \underbrace{\phi_1 \rightarrow (\phi_2 \rightarrow (\ldots(\phi_n \rightarrow \psi)\ldots))}_{\vDash \eta}$ prove $\vdash \underbrace{\phi_1 \rightarrow (\phi_2 \rightarrow (\ldots(\phi_n \rightarrow \psi)\ldots))}_{\vdash \eta}$

From $\vDash \eta$ prove $\vdash \eta$

Proof for Completeness of ND for Prop Logic

Proof for Step 2: From $\vDash \eta$ prove $\vdash \eta$.

Proof Idea: Sub-proof for every line in truth table

- Assuming $\vDash \eta$ holds. Let $p_1, \ldots p_n$ the propositional atoms of η
- We know that η evaluates to **true** for all $\mathbf{2^n}$ **lines** of the truth table
- Thus, we can encode **each line in truth table as sequent** and know that the sequent is correct.
  - This step is proven by Proposition 1.38, page 51, book: Logic in Computer Science

Proof for Completeness of ND for Prop Logic

Proof for Step 2: From $\vDash \eta$ prove $\vdash \eta$.

Proof Idea: Sub-proof for every line in truth table

- Assuming $\vDash \eta$ holds. Let $p_1, \ldots p_n$ the propositional atoms of η
- We know that η evaluates to **true** for all **2^n lines** of the truth table
- Thus, we can encode **each line in truth table as sequent** and know that the sequent is correct.
  - $\neg p_n, \ldots \neg p_2, \neg p_1 \vdash \eta$
  - $\neg p_n, \ldots \neg p_2, \quad p_1 \vdash \eta$
  - $\neg p_n, \ldots \quad p_2, \neg p_1 \vdash \eta$
  - …
  - $p_n, \ldots \quad p_2, \quad p_1 \vdash \eta$

Proof for Completeness of ND for Prop Logic

Proof for Step 2: From $\vDash \eta$ prove $\vdash \eta$.

Proof Idea: Sub-proof for every line in truth table

- **Combine proofs into single proof** without premises
  - → Use LEM for all propositional atoms, then separately assume all cases
  - Example: How to do this for $\vdash (p \wedge q) \to p$

1	$p \vee \neg p$							LEM
2	p			ass	$\neg p$			ass
3	$q \vee \neg q$			LEM	$q \vee \neg q$			LEM
4	q	ass	$\neg q$	ass	q	ass	$\neg q$	ass
5	$\vdots$		$\vdots$		$\vdots$		$\vdots$	
6								
7	$p \wedge q \to p$		$p \wedge q \to p$		$p \wedge q \to p$		$p \wedge q \to p$	
8	$p \wedge q \to p$			$\vee$e	$p \wedge q \to p$			$\vee$e
9	$p \wedge q \to p$							$\vee$e

Proof for Completeness of ND for Prop Logic

Proof Idea:

- Assuming that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds:
- Step 1: We show that $\vDash \phi_1 \to \big(\phi_2 \to (\ldots(\phi_n \to \psi)\ldots)\big)$ holds. ✓
- **Step 2: We show that $\vdash \boldsymbol{\phi_1 \to \big(\phi_2 \to (\ldots(\phi_n \to \psi)\ldots)\big)}$ holds.** ✓
- Step 3: Finally, we show that $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ holds. ✓

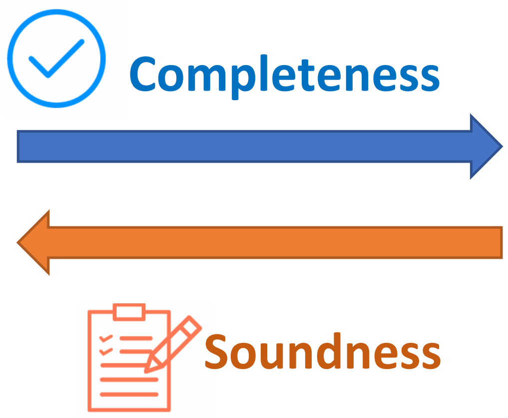
We have proven the **Completeness Theorem**

- „If $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ holds."

Soundness and Completeness

We have proven that **Natural Deduction** for prop. logic is **sound** and **complete!**

- $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds **if and only if** $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ holds.



Outline

- Recap: Mathematical Induction
- Prove that ND for prop. logic is **sound**
- Prove that ND for prop. logic is **complete**
- Prove tautologies with uniform method
  - from completeness proof

Uniform Approach To Prove Tautologies

- Use **LEM** for all propositional atoms, then **separately assume all cases**
  - Proof contains sub-proof for each line in truth table
- Example: $\vdash (p \land q) \to p$

1	$p \lor \neg p$							LEM
2	p			ass	$\neg p$			ass
3	$q \lor \neg q$			LEM	$q \lor \neg q$			LEM
4	q	ass	$\neg q$	ass	q	ass	$\neg q$	ass
5	$p \land q$	ass	$p \land q$	ass	$p \land q$	ass	$p \land q$	ass
6	p	$\land e$	p	$\land e$	p	$\land e$	p	$\land e$
7	$p \land q \to p$	$\to i$	$p \land q \to p$	$\to i$	$p \land q \to p$	$\to i$	$p \land q \to p$	$\to i$
8	$p \land q \to p$			$\lor$e	$p \land q \to p$			$\lor$e
9	$p \land q \to p$							$\lor$e

Thank You