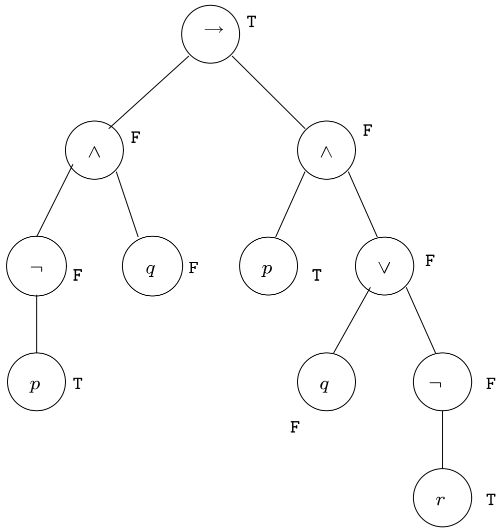

Figure 1.7. The evaluation of a logical formula under a given valuation.

p	q	$\neg p$	$\neg q$	$p \to \neg q$	$q \vee \neg p$	$(p \to \neg q) \to (q \vee \neg p)$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Figure 1.8. An example of a truth table for a more complex logical formula.

Finally, column 7 results from applying the truth table of $\to$ to columns 5 and 6.

1.4.2 Mathematical induction

Here is a little anecdote about the German mathematician Gauss who, as a pupil at age 8, did not pay attention in class (can you imagine?), with the result that his teacher made him sum up all natural numbers from 1 to 100. The story has it that Gauss came up with the correct answer 5050 within seconds, which infuriated his teacher. How did Gauss do it? Well, possibly he knew that

$$1 + 2 + 3 + 4 + \cdots + n = \frac{n \cdot (n+1)}{2} \tag{1.5}$$

for all natural numbers n.[9] Thus, taking $n = 100$, Gauss could easily calculate:

$$1 + 2 + 3 + 4 + \cdots + 100 = \frac{100 \cdot 101}{2} = 5050.$$

Mathematical induction allows us to prove equations, such as the one in (1.5), for arbitrary n. More generally, it allows us to show that *every* natural number satisfies a certain property. Suppose we have a property M which we think is true of all natural numbers. We write $M(5)$ to say that the property is true of 5, etc. Suppose that we know the following two things about the property M:

1. **Base case:** The natural number 1 has property M, i.e. we have a proof of $M(1)$.
2. **Inductive step:** If n is a natural number which *we assume* to have property $M(n)$, then *we can show* that $n + 1$ has property $M(n + 1)$; i.e. we have a proof of $M(n) \to M(n + 1)$.

Definition 1.30 The principle of mathematical induction says that, on the grounds of these two pieces of information above, every natural number n has property $M(n)$. The assumption of $M(n)$ in the inductive step is called the *induction hypothesis*.

Why does this principle make sense? Well, take *any* natural number k. If k equals 1, then k has property $M(1)$ using the base case and so we are done. Otherwise, we can use the inductive step, applied to $n = 1$, to infer that $2 = 1 + 1$ has property $M(2)$. We can do that using $\to$e, for we know that 1 has the property in question. Now we use that same inductive step on $n = 2$ to infer that 3 has property $M(3)$ and we repeat this until we reach $n = k$ (see Figure 1.9). Therefore, we should have no objections about using the principle of mathematical induction for natural numbers.

Returning to Gauss' example we claim that the sum $1 + 2 + 3 + 4 + \cdots + n$ equals $n \cdot (n + 1)/2$ for all natural numbers n.

Theorem 1.31 *The sum* $1 + 2 + 3 + 4 + \cdots + n$ *equals* $n \cdot (n + 1)/2$ *for all natural numbers* n.

[9] There is another way of finding the sum $1 + 2 + \cdots + 100$, which works like this: write the sum backwards, as $100 + 99 + \cdots + 1$. Now add the forwards and backwards versions, obtaining $101 + 101 + \cdots + 101$ (100 times), which is 10100. Since we added the sum to itself, we now divide by two to get the answer 5050. Gauss probably used this method; but the method of mathematical induction that we explore in this section is much more powerful and can be applied in a wide variety of situations.

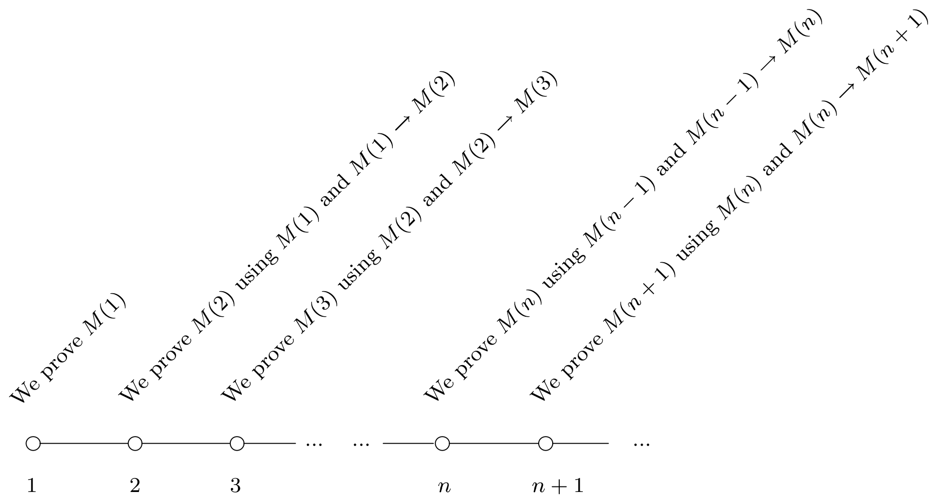

Figure 1.9. How the principle of mathematical induction works. By proving just two facts, $M(1)$ and $M(n) \to M(n+1)$ for a formal (and unconstrained) parameter n, we are able to deduce $M(k)$ for each natural number k.

PROOF: We use mathematical induction. In order to reveal the fine structure of our proof we write LHS_n for the expression $1 + 2 + 3 + 4 + \cdots + n$ and RHS_n for $n \cdot (n+1)/2$. Thus, we need to show $\mathrm{LHS}_n = \mathrm{RHS}_n$ for all $n \geq 1$.

Base case: If n equals 1, then LHS_1 is just 1 (there is only one summand), which happens to equal $\mathrm{RHS}_1 = 1 \cdot (1+1)/2$.

Inductive step: Let us assume that $\mathrm{LHS}_n = \mathrm{RHS}_n$. Recall that this assumption is called the induction hypothesis; it is the driving force of our argument. We need to show $\mathrm{LHS}_{n+1} = \mathrm{RHS}_{n+1}$, i.e. that the longer sum $1 + 2 + 3 + 4 + \cdots + (n+1)$ equals $(n+1) \cdot ((n+1)+1)/2$. The key observation is that the sum $1 + 2 + 3 + 4 + \cdots + (n+1)$ is nothing but the sum $(1 + 2 + 3 + 4 + \cdots + n) + (n+1)$ of two summands, where the first one is the sum of our induction hypothesis. The latter says that $1 + 2 + 3 + 4 + \cdots + n$ equals $n \cdot (n+1)/2$, and we are certainly entitled to substitute equals for equals in our reasoning. Thus, we compute

$$\begin{aligned}
&\mathrm{LHS}_{n+1} \\
&\quad = 1 + 2 + 3 + 4 + \cdots + (n+1) \\
&\quad = \mathrm{LHS}_n + (n+1) \quad \text{regrouping the sum}
\end{aligned}$$

$$
\begin{aligned}
&= \text{RHS}_n + (n+1) \quad \text{by our induction hypothesis} \\
&= \tfrac{n\cdot(n+1)}{2} + (n+1) \\
&= \tfrac{n\cdot(n+1)}{2} + \tfrac{2\cdot(n+1)}{2} \quad \text{arithmetic} \\
&= \tfrac{(n+2)\cdot(n+1)}{2} \quad \text{arithmetic} \\
&= \tfrac{((n+1)+1)\cdot(n+1)}{2} \quad \text{arithmetic} \\
&= \text{RHS}_{n+1}.
\end{aligned}
$$

Since we successfully showed the base case and the inductive step, we can use mathematical induction to infer that all natural numbers n have the property stated in the theorem above. □

Actually, there are numerous variations of this principle. For example, we can think of a version in which the base case is $n = 0$, which would then cover all natural numbers including 0. Some statements hold only for all natural numbers, say, greater than 3. So you would have to deal with a base case 4, but keep the version of the inductive step (see the exercises for such an example). The use of mathematical induction typically suceeds on properties $M(n)$ that involve inductive definitions (e.g. the definition of k^l with $l \geq 0$). Sentence (3) on page 2 suggests there may be true properties $M(n)$ for which mathematical induction won't work.

Course-of-values induction. There is a variant of mathematical induction in which the induction hypothesis for proving $M(n+1)$ is not just $M(n)$, but the conjunction $M(1) \land M(2) \land \cdots \land M(n)$. In that variant, called *course-of-values* induction, there doesn't have to be an explicit base case at all – everything can be done in the inductive step.

How can this work without a base case? The answer is that the base case is implicitly included in the inductive step. Consider the case $n = 3$: the inductive-step instance is $M(1) \land M(2) \land M(3) \to M(4)$. Now consider $n = 1$: the inductive-step instance is $M(1) \to M(2)$. What about the case when n equals 0? In this case, there are zero formulas on the left of the $\to$, so we have to prove $M(1)$ from nothing at all. The inductive-step instance is simply the obligation to show $M(1)$. You might find it useful to modify Figure 1.9 for course-of-values induction.

Having said that the base case is implicit in course-of-values induction, it frequently turns out that it still demands special attention when you get inside trying to prove the inductive case. We will see precisely this in the two applications of course-of-values induction in the following pages.

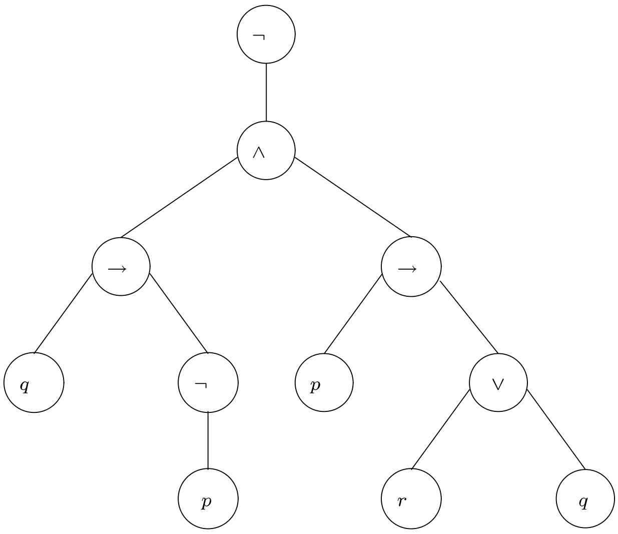

Figure 1.10. A parse tree with height 5.

In computer science, we often deal with finite structures of some kind, data structures, programs, files etc. Often we need to show that *every* instance of such a structure has a certain property. For example, the well-formed formulas of Definition 1.27 have the property that the number of '(' brackets in a particular formula equals its number of ')' brackets. We can use mathematical induction on the domain of natural numbers to prove this. In order to succeed, we somehow need to connect well-formed formulas to natural numbers.

Definition 1.32 Given a well-formed formula ϕ, we define its *height* to be 1 plus the length of the longest path of its parse tree.

For example, consider the well-formed formulas in Figures 1.3, 1.4 and 1.10. Their heights are 5, 6 and 5, respectively. In Figure 1.3, the longest path goes from $\rightarrow$ to $\wedge$ to $\vee$ to $\neg$ to r, a path of length 4, so the height is $4 + 1 = 5$. Note that the height of atoms is $1 + 0 = 1$. Since every well-formed formula has finite height, we can show statements about all well-formed formulas by mathematical induction on their height. This trick is most often called *structural induction*, an important reasoning technique in computer science. Using the notion of the height of a parse tree, we realise that structural induction is just a special case of course-of-values induction.

Theorem 1.33 *For every well-formed propositional logic formula, the number of left brackets is equal to the number of right brackets.*

PROOF: We proceed by course-of-values induction on the height of well-formed formulas ϕ. Let $M(n)$ mean 'All formulas of height n have the same number of left and right brackets.' We assume $M(k)$ for each $k < n$ and try to prove $M(n)$. Take a formula ϕ of height n.

- **Base case:** Then $n = 1$. This means that ϕ is just a propositional atom. So there are no left or right brackets, 0 equals 0.
- **Course-of-values inductive step:** Then $n > 1$ and so the root of the parse tree of ϕ must be $\neg$, $\rightarrow$, $\vee$ or $\wedge$, for ϕ is well-formed. We assume that it is $\rightarrow$, the other three cases are argued in a similar way. Then ϕ equals $(\phi_1 \rightarrow \phi_2)$ for some well-formed formulas ϕ_1 and ϕ_2 (of course, they are just the left, respectively right, linear representations of the root's two subtrees). It is clear that the heights of ϕ_1 and ϕ_2 are strictly smaller than n. Using the induction hypothesis, we therefore conclude that ϕ_1 has the same number of left and right brackets and that the same is true for ϕ_2. But in $(\phi_1 \rightarrow \phi_2)$ we added just two more brackets, one '(' and one ')'. Thus, the number of occurrences of '(' and ')' in ϕ is the same. □

The formula $(p \rightarrow (q \wedge \neg r))$ illustrates why we could not prove the above directly with mathematical induction on the height of formulas. While this formula has height 4, its two subtrees have heights 1 and 3, respectively. Thus, an induction hypothesis for height 3 would have worked for the right subtree but failed for the left subtree.

1.4.3 Soundness of propositional logic

The natural deduction rules make it possible for us to develop rigorous threads of argumentation, in the course of which we arrive at a conclusion ψ assuming certain other propositions $\phi_1, \phi_2, \ldots, \phi_n$. In that case, we said that the sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid. Do we have any evidence that these rules are all *correct* in the sense that valid sequents all 'preserve truth' computed by our truth-table semantics?

Given a proof of $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$, is it conceivable that there is a valuation in which ψ above is false although all propositions $\phi_1, \phi_2, \ldots, \phi_n$ are true? Fortunately, this is not the case and in this subsection we demonstrate why this is so. Let us suppose that some proof in our natural deduction calculus has established that the sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid. We need to show: for all valuations in which all propositions $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to T, ψ evaluates to T as well.

Definition 1.34 If, for all valuations in which all $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to T, ψ evaluates to T as well, we say that

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds and call $\vDash$ the *semantic entailment* relation.

Let us look at some examples of this notion.

1. Does $p \wedge q \vDash p$ hold? Well, we have to inspect all assignments of truth values to p and q; there are four of these. Whenever such an assignment computes T for $p \wedge q$ we need to make sure that p is true as well. But $p \wedge q$ computes T only if p and q are true, so $p \wedge q \vDash p$ is indeed the case.
2. What about the relationship $p \vee q \vDash p$? There are three assignments for which $p \vee q$ computes T, so p would have to be true for all of these. However, if we assign T to q and F to p, then $p \vee q$ computes T, but p is false. Thus, $p \vee q \vDash p$ does not hold.
3. What if we modify the above to $\neg q, p \vee q \vDash p$? Notice that we have to be concerned only about valuations in which $\neg q$ *and* $p \vee q$ evaluate to T. This forces q to be false, which in turn forces p to be true. Hence $\neg q, p \vee q \vDash p$ is the case.
4. Note that $p \vDash q \vee \neg q$ holds, despite the fact that no atomic proposition on the right of $\vDash$ occurs on the left of $\vDash$.

From the discussion above we realize that a soundness argument has to show: if $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.

Theorem 1.35 (Soundness) *Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional logic formulas. If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.*

PROOF: Since $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid we know there is a proof of ψ from the premises $\phi_1, \phi_2, \ldots, \phi_n$. We now do a pretty slick thing, namely, we reason by *mathematical induction on the length of this proof!* The length of a proof is just the number of lines it involves. So let us be perfectly clear about what it is we mean to show. We intend to show the assertion $M(k)$:

> *'For all sequents $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ $(n \geq 0)$ which have a proof of length k, it is the case that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.'*

by course-of-values induction on the natural number k. This idea requires some work, though. The sequent $p \wedge q \to r \vdash p \to (q \to r)$ has a proof

1	$p \wedge q \to r$	premise
2	p	assumption
3	q	assumption
4	$p \wedge q$	$\wedge$i 2, 3
5	r	$\to$e 1, 4
6	$q \to r$	$\to$i 3–5
7	$p \to (q \to r)$	$\to$i 2–6

but if we remove the last line or several of the last lines, we no longer have a proof as the outermost box does not get closed. We get a complete proof, though, by removing the last line and re-writing the assumption of the outermost box as a premise:

1	$p \wedge q \to r$	premise
2	p	premise
3	q	assumption
4	$p \wedge q$	$\wedge$i 2, 3
5	r	$\to$e 1, 4
6	$q \to r$	$\to$i 3–5

This is a proof of the sequent $p \wedge q \to r,\ p \vdash p \to r$. The induction hypothesis then ensures that $p \wedge q \to r,\ p \vDash p \to r$ holds. But then we can also reason that $p \wedge q \to r \vDash p \to (q \to r)$ holds as well – why?

Let's proceed with our proof by induction. We assume $M(k')$ for each $k' < k$ and we try to prove $M(k)$.

Base case: a one-line proof. If the proof has length 1 ($k = 1$), then it must be of the form

1	ϕ	premise

since all other rules involve more than one line. This is the case when $n = 1$ and ϕ_1 and ψ equal ϕ, i.e. we are dealing with the sequent $\phi \vdash \phi$. Of course, since ϕ evaluates to T so does ϕ. Thus, $\phi \vDash \phi$ holds as claimed.

Course-of-values inductive step: Let us assume that the proof of the sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ has length k and that the statement we want to prove is true for all numbers less than k. Our proof has the following structure:

$$\begin{array}{ll} 1 & \phi_1 \text{ premise} \\ 2 & \phi_2 \text{ premise} \\ & \vdots \\ n & \phi_n \text{ premise} \\ & \vdots \\ k & \psi \text{ justification} \end{array}$$

There are two things we don't know at this point. First, what is happening in between those dots? Second, what was the last rule applied, i.e. what is the justification of the last line? The first uncertainty is of no concern; this is where mathematical induction demonstrates its power. The second lack of knowledge is where all the work sits. In this generality, there is simply no way of knowing which rule was applied last, so we need to consider all such rules in turn.

1. Let us suppose that this last rule is $\wedge$i. Then we know that ψ is of the form $\psi_1 \wedge \psi_2$ and the justification in line k refers to two lines further up which have ψ_1, respectively ψ_2, as their conclusions. Suppose that these lines are k_1 and k_2. Since k_1 and k_2 are smaller than k, we see that there exist proofs of the sequents $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1$ and $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_2$ with length *less than* k – just take the first k_1, respectively k_2, lines of our original proof. Using the induction hypothesis, we conclude that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi_1$ and $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi_2$ holds. But these two relations imply that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi_1 \wedge \psi_2$ holds as well – why?
2. If ψ has been shown using the rule $\vee$e, then we must have proved, assumed or given as a premise some formula $\eta_1 \vee \eta_2$ in some line k' with $k' < k$, which was referred to via $\vee$e in the justification of line k. Thus, we have a shorter proof of the sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \eta_1 \vee \eta_2$ within that proof, obtained by turning all assumptions of boxes that are open at line k' into premises. In a similar way we obtain proofs of the sequents $\phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vdash \psi$ and $\phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vdash \psi$ from the case analysis of $\vee$e. By our induction hypothesis, we conclude that the relations $\phi_1, \phi_2, \ldots, \phi_n \vDash \eta_1 \vee \eta_2$, $\phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vDash \psi$ and $\phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vDash \psi$ hold. But together these three relations then force that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds as well – why?
3. You can guess by now that the rest of the argument checks each possible proof rule in turn and ultimately boils down to verifying that our natural deduction

rules behave semantically in the same way as their corresponding truth tables evaluate. We leave the details as an exercise. □

The soundness of propositional logic is useful in ensuring the *non-existence* of a proof for a given sequent. Let's say you try to prove that $\phi_1, \phi_2, \ldots, \phi_2 \vdash \psi$ is valid, but that your best efforts won't succeed. How could you be sure that no such proof can be found? After all, it might just be that you can't find a proof even though there is one. It suffices to find a valuation in which ϕ_i evaluate to T whereas ψ evaluates to F. Then, by definition of $\vDash$, we don't have $\phi_1, \phi_2, \ldots, \phi_2 \vDash \psi$. Using soundness, this means that $\phi_1, \phi_2, \ldots, \phi_2 \vdash \psi$ cannot be valid. Therefore, this sequent does not have a proof. You will practice this method in the exercises.

1.4.4 Completeness of propositional logic

In this subsection, we hope to convince you that the natural deduction rules of propositional logic are *complete*: whenever $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds, then there exists a natural deduction proof for the sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$. Combined with the soundness result of the previous subsection, we then obtain

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid iff } \phi_1, \phi_2, \ldots, \phi_n \vDash \psi \text{ holds.}$$

This gives you a certain freedom regarding which method you prefer to use. Often it is much easier to show one of these two relationships (although neither of the two is universally better, or easier, to establish). The first method involves a *proof search*, upon which the *logic programming* paradigm is based. The second method typically forces you to compute a truth table which is exponential in the size of occurring propositional atoms. Both methods are intractable in general but particular instances of formulas often respond differently to treatment under these two methods.

The remainder of this section is concerned with an argument saying that if $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds, then $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid. Assuming that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds, the argument proceeds in three steps:

Step 1: We show that $\vDash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots (\phi_n \to \psi) \ldots)))$ holds.
Step 2: We show that $\vdash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots (\phi_n \to \psi) \ldots)))$ is valid.
Step 3: Finally, we show that $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid.

The first and third steps are quite easy; all the real work is done in the second one.

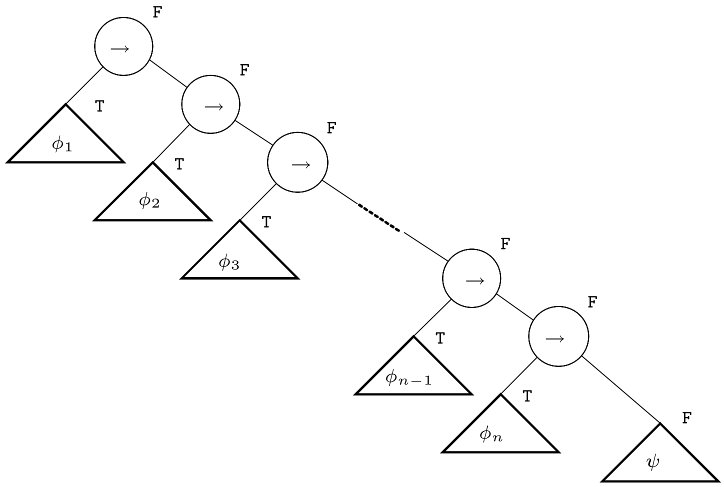

Figure 1.11. The only way this parse tree can evaluate to F. We represent parse trees for ϕ_1, ϕ_2, ..., ϕ_n as triangles as their internal structure does not concern us here.

Step 1:

Definition 1.36 A formula of propositional logic ϕ is called a *tautology* iff it evaluates to T under all its valuations, i.e. iff $\vDash \phi$.

Supposing that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds, let us verify that $\phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots(\phi_n \to \psi)\ldots)))$ is indeed a tautology. Since the latter formula is a nested implication, it can evaluate to F only if all $\phi_1, \phi_2, .., \phi_n$ evaluate to T *and* ψ evaluates to F; see its parse tree in Figure 1.11. But this contradicts the fact that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds. Thus, $\vDash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots(\phi_n \to \psi)\ldots)))$ holds.

Step 2:

Theorem 1.37 *If $\vDash \eta$ holds, then $\vdash \eta$ is valid. In other words, if η is a tautology, then η is a theorem.*

This step is the hard one. Assume that $\vDash \eta$ holds. Given that η contains n distinct propositional atoms $p_1, p_2, \ldots, p_n$ we know that η evaluates to T for all 2^n lines in its truth table. (Each line lists a valuation of η.) How can we use this information to construct a proof for η? In some cases this can be done quite easily by taking a very good look at the concrete structure of η. But here we somehow have to come up with a *uniform* way of building such a proof. The key insight is to 'encode' each line in the truth table of η

as a sequent. Then we construct proofs for these 2^n sequents and assemble them into a proof of η.

Proposition 1.38 *Let ϕ be a formula such that $p_1, p_2, \dots, p_n$ are its only propositional atoms. Let l be any line number in ϕ's truth table. For all $1 \le i \le n$ let $\hat{p}_i$ be p_i if the entry in line l of p_i is* T*, otherwise $\hat{p}_i$ is $\neg p_i$. Then we have*

1. *$\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \phi$ is provable if the entry for ϕ in line l is* T
2. *$\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \neg\phi$ is provable if the entry for ϕ in line l is* F

PROOF: This proof is done by structural induction on the formula ϕ, that is, mathematical induction on the height of the parse tree of ϕ.

1. If ϕ is a propositional atom p, we need to show that $p \vdash p$ and $\neg p \vdash \neg p$. These have one-line proofs.
2. If ϕ is of the form $\neg\phi_1$ we again have two cases to consider. First, assume that ϕ evaluates to T. In this case ϕ_1 evaluates to F. Note that ϕ_1 has the same atomic propositions as ϕ. We may use the induction hypothesis on ϕ_1 to conclude that $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \neg\phi_1$; but $\neg\phi_1$ is just ϕ, so we are done.
Second, if ϕ evaluates to F, then ϕ_1 evaluates to T and we get $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \phi_1$ by induction. Using the rule $\neg\neg$i, we may extend the proof of $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \phi_1$ to one for $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \neg\neg\phi_1$; but $\neg\neg\phi_1$ is just $\neg\phi$, so again we are done.

The remaining cases all deal with two subformulas: ϕ equals $\phi_1 \circ \phi_2$, where $\circ$ is $\to$, $\wedge$ or $\vee$. In all these cases let $q_1, \dots, q_l$ be the propositional atoms of ϕ_1 and $r_1, \dots, r_k$ be the propositional atoms of ϕ_2. Then we certainly have $\{q_1, \dots, q_l\} \cup \{r_1, \dots, r_k\} = \{p_1, \dots, p_n\}$. Therefore, whenever $\hat{q}_1, \dots, \hat{q}_l \vdash \psi_1$ and $\hat{r}_1, \dots, \hat{r}_k \vdash \psi_2$ are valid so is $\hat{p}_1, \dots, \hat{p}_n \vdash \psi_1 \wedge \psi_2$ using the rule $\wedge$i. In this way, we can use our induction hypothesis and only owe proofs that the conjunctions we conclude allow us to prove the desired conclusion for ϕ or $\neg\phi$ as the case may be.

3. To wit, let ϕ be $\phi_1 \to \phi_2$. If ϕ evaluates to F, then we know that ϕ_1 evaluates to T and ϕ_2 to F. Using our induction hypothesis, we have $\hat{q}_1, \dots, \hat{q}_l \vdash \phi_1$ and $\hat{r}_1, \dots, \hat{r}_k \vdash \neg\phi_2$, so $\hat{p}_1, \dots, \hat{p}_n \vdash \phi_1 \wedge \neg\phi_2$ follows. We need to show $\hat{p}_1, \dots, \hat{p}_n \vdash \neg(\phi_1 \to \phi_2)$; but using $\hat{p}_1, \dots, \hat{p}_n \vdash \phi_1 \wedge \neg\phi_2$, this amounts to proving the sequent $\phi_1 \wedge \neg\phi_2 \vdash \neg(\phi_1 \to \phi_2)$, which we leave as an exercise.
If ϕ evaluates to T, then we have three cases. First, if ϕ_1 evaluates to F and ϕ_2 to F, then we get, by our induction hypothesis, that $\hat{q}_1, \dots, \hat{q}_l \vdash \neg\phi_1$ and $\hat{r}_1, \dots, \hat{r}_k \vdash \neg\phi_2$, so $\hat{p}_1, \dots, \hat{p}_n \vdash \neg\phi_1 \wedge \neg\phi_2$ follows. Again, we need only to show the sequent $\neg\phi_1 \wedge \neg\phi_2 \vdash \phi_1 \to \phi_2$, which we leave as an exercise. Second, if ϕ_1 evaluates to F and ϕ_2 to T, we use our induction hypothesis to arrive at

$\hat{p}_1, \dots, \hat{p}_n \vdash \neg\phi_1 \wedge \phi_2$ and have to prove $\neg\phi_1 \wedge \phi_2 \vdash \phi_1 \to \phi_2$, which we leave as an exercise. Third, if ϕ_1 and ϕ_2 evaluate to T, we arrive at $\hat{p}_1, \dots, \hat{p}_n \vdash \phi_1 \wedge \phi_2$, using our induction hypothesis, and need to prove $\phi_1 \wedge \phi_2 \vdash \phi_1 \to \phi_2$, which we leave as an exercise as well.

4. If ϕ is of the form $\phi_1 \wedge \phi_2$, we are again dealing with four cases in total. First, if ϕ_1 and ϕ_2 evaluate to T, we get $\hat{q}_1, \dots, \hat{q}_l \vdash \phi_1$ and $\hat{r}_1, \dots, \hat{r}_k \vdash \phi_2$ by our induction hypothesis, so $\hat{p}_1, \dots, \hat{p}_n \vdash \phi_1 \wedge \phi_2$ follows. Second, if ϕ_1 evaluates to F and ϕ_2 to T, then we get $\hat{p}_1, \dots, \hat{p}_n \vdash \neg\phi_1 \wedge \phi_2$ using our induction hypothesis and the rule $\wedge$i as above and we need to prove $\neg\phi_1 \wedge \phi_2 \vdash \neg(\phi_1 \wedge \phi_2)$, which we leave as an exercise. Third, if ϕ_1 and ϕ_2 evaluate to F, then our induction hypothesis and the rule $\wedge$i let us infer that $\hat{p}_1, \dots, \hat{p}_n \vdash \neg\phi_1 \wedge \neg\phi_2$; so we are left with proving $\neg\phi_1 \wedge \neg\phi_2 \vdash \neg(\phi_1 \wedge \phi_2)$, which we leave as an exercise. Fourth, if ϕ_1 evaluates to T and ϕ_2 to F, we obtain $\hat{p}_1, \dots, \hat{p}_n \vdash \phi_1 \wedge \neg\phi_2$ by our induction hypothesis and we have to show $\phi_1 \wedge \neg\phi_2 \vdash \neg(\phi_1 \wedge \phi_2)$, which we leave as an exercise.
5. Finally, if ϕ is a disjunction $\phi_1 \vee \phi_2$, we again have four cases. First, if ϕ_1 and ϕ_2 evaluate to F, then our induction hypothesis and the rule $\wedge$i give us $\hat{p}_1, \dots, \hat{p}_n \vdash \neg\phi_1 \wedge \neg\phi_2$ and we have to show $\neg\phi_1 \wedge \neg\phi_2 \vdash \neg(\phi_1 \vee \phi_2)$, which we leave as an exercise. Second, if ϕ_1 and ϕ_2 evaluate to T, then we obtain $\hat{p}_1, \dots, \hat{p}_n \vdash \phi_1 \wedge \phi_2$, by our induction hypothesis, and we need a proof for $\phi_1 \wedge \phi_2 \vdash \phi_1 \vee \phi_2$, which we leave as an exercise. Third, if ϕ_1 evaluates to F and ϕ_2 to T, then we arrive at $\hat{p}_1, \dots, \hat{p}_n \vdash \neg\phi_1 \wedge \phi_2$, using our induction hypothesis, and need to establish $\neg\phi_1 \wedge \phi_2 \vdash \phi_1 \vee \phi_2$, which we leave as an exercise. Fourth, if ϕ_1 evaluates to T and ϕ_2 to F, then $\hat{p}_1, \dots, \hat{p}_n \vdash \phi_1 \wedge \neg\phi_2$ results from our induction hypothesis and all we need is a proof for $\phi_1 \wedge \neg\phi_2 \vdash \phi_1 \vee \phi_2$, which we leave as an exercise. □

We apply this technique to the formula $\vDash \phi_1 \to (\phi_2 \to (\phi_3 \to (\dots(\phi_n \to \psi)\dots)))$. Since it is a tautology it evaluates to T in all 2^n lines of its truth table; thus, the proposition above gives us 2^n many proofs of $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \eta$, one for each of the cases that $\hat{p}_i$ is p_i or $\neg p_i$. Our job now is to assemble all these proofs into a single proof for η which does not use any premises. We illustrate how to do this for an example, the tautology $p \wedge q \to p$.

The formula $p \wedge q \to p$ has two propositional atoms p and q. By the proposition above, we are guaranteed to have a proof for each of the four sequents

$$p, q \vdash p \wedge q \to p$$
$$\neg p, q \vdash p \wedge q \to p$$
$$p, \neg q \vdash p \wedge q \to p$$
$$\neg p, \neg q \vdash p \wedge q \to p.$$

Ultimately, we want to prove $p \wedge q \to p$ by appealing to the four proofs of the sequents above. Thus, we somehow need to get rid of the premises on the left-hand sides of these four sequents. This is the place where we rely on the law of the excluded middle which states $r \vee \neg r$, for any r. We use LEM for all propositional atoms (here p and q) and then we separately assume all the four cases, by using $\vee$e. That way we can invoke all four proofs of the sequents above and use the rule $\vee$e repeatedly until we have got rid of all our premises. We spell out the combination of these four phases schematically:

1	$p \vee \neg p$							LEM
2	p			ass	$\neg p$			ass
3	$q \vee \neg q$			LEM	$q \vee \neg q$			LEM
4	q	ass	$\neg q$	ass	q	ass	$\neg q$	ass
5	$\vdots$		$\vdots$		$\vdots$		$\vdots$	
6								
7	$p \wedge q \to p$		$p \wedge q \to p$		$p \wedge q \to p$		$p \wedge q \to p$	
8	$p \wedge q \to p$			$\vee$e	$p \wedge q \to p$			$\vee$e
9	$p \wedge q \to p$							$\vee$e

As soon as you understand how this particular example works, you will also realise that it will work for an arbitrary tautology with n distinct atoms. Of course, it seems ridiculous to prove $p \wedge q \to p$ using a proof that is this long. But remember that this illustrates a *uniform* method that constructs a proof for every tautology η, no matter how complicated it is.

Step 3: Finally, we need to find a proof for $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$. Take the proof for $\vdash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots(\phi_n \to \psi)\ldots)))$ given by step 2 and augment its proof by introducing $\phi_1, \phi_2, \ldots, \phi_n$ as premises. Then apply $\to$e n times on each of these premises (starting with ϕ_1, continuing with ϕ_2 etc.). Thus, we arrive at the conclusion ψ which gives us a proof for the sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$.

Corollary 1.39 (Soundness and Completeness) *Let* $\phi_1, \phi_2, \ldots, \phi_n, \psi$ *be formulas of propositional logic. Then* $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ *is holds iff the sequent* $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ *is valid.*

1.5 Normal forms

In the last section, we showed that our proof system for propositional logic is sound and complete for the truth-table semantics of formulas in Figure 1.6.