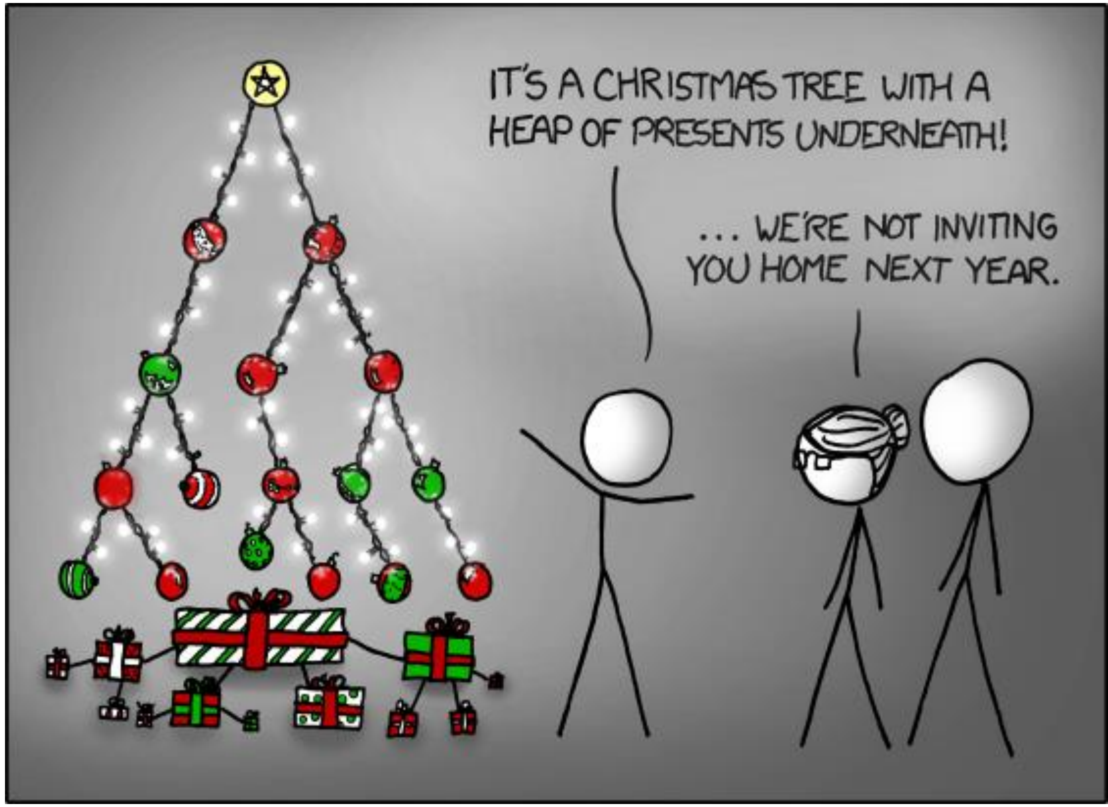
Logic and Computability

Binary Decision Diagrams (BDDs)



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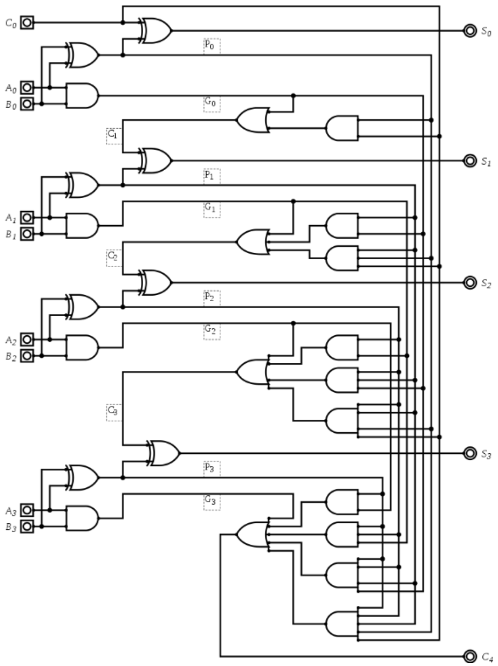
Motivation – BDDs

- Formulas are huge
  - E.g., when presenting circuit as formula
  - Hundreds of thousands of variables, millions of clauses….
- We need efficient methods
  - to store, to manipulate formula, and to decide formulas

Some Circuit



Automatically Translate

Nr variables **Nr clauses**

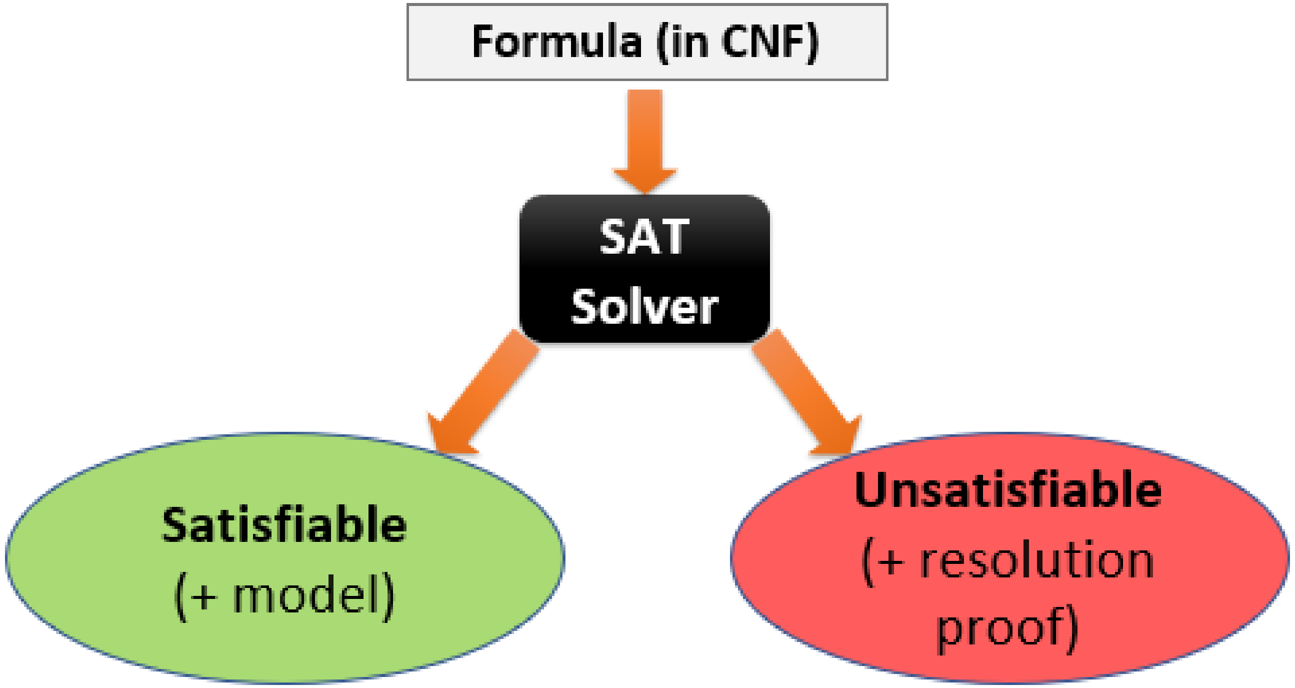
```
p cnf 51639 368352
-1 7 0
-1 6 0
-1 5 0
-1 -4 0
-1 3 0
-1 2 0
-1 -8 0
-9 15 0
-9 14 0
-9 13 0
-9 -12 0
-9 11 0
-9 10 0
-9 -16 0
-17 23 0
-17 22 0
```

$(\neg x_1 \vee x_7) \wedge$

$(\neg x_1 \vee x_6) \wedge \cdots$

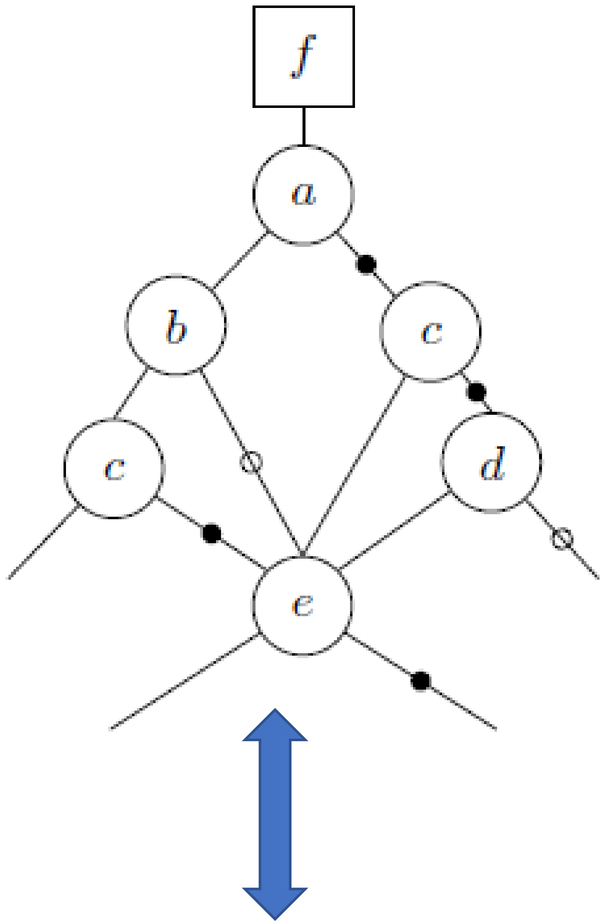
Motivation – BDDs

- Last week
  - SAT Solvers
  - DPLL - Efficient algorithm to decide huge formulas



Motivation – BDDs

- This week- BDDs
  - Graph-based data structure
  - To represent and manipulate formulas

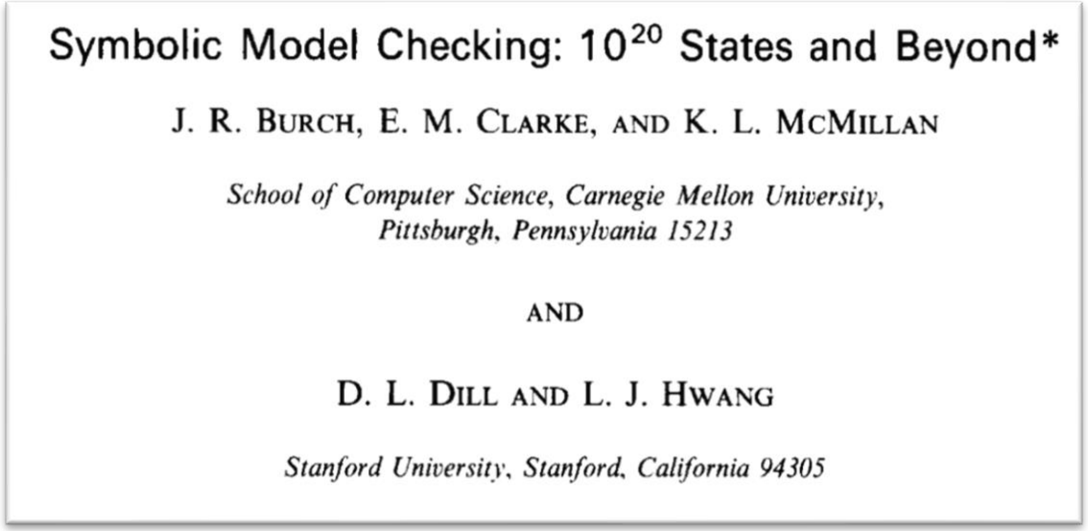


$$f = (a \wedge b \wedge c) \vee (a \wedge \neg b \wedge e) \vee (a \wedge b \wedge \neg c \wedge \neg e) \vee (\neg a \wedge \neg c \wedge d) \vee (\neg a \wedge c \wedge \neg e) \vee (\neg a \wedge \neg c \wedge d \wedge e)$$

Jerry R. Burch, Edmund M. Clarke, Kenneth L. McMillan, David L. Dill, L. J. Hwang:
Symbolic Model Checking: 10^20 States and Beyond. LICS 1990: 428-439

Motivation – BDDs

- This week- BDDs
  - Graph-based data structure
  - To represent and manipulate formulas
  - E.g., Used in hardware and software verification tools



Symbolic Model Checking: 10^{20} States and Beyond*

J. R. BURCH, E. M. CLARKE, AND K. L. MCMILLAN

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AND

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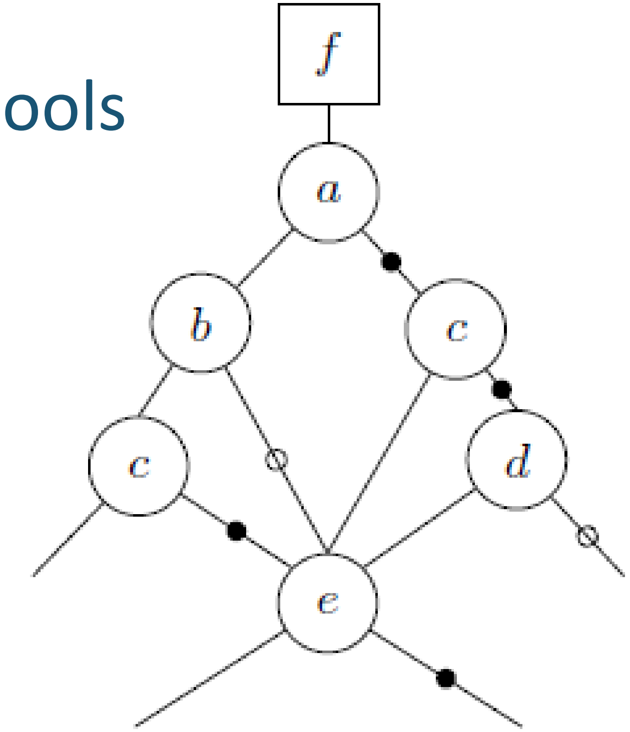

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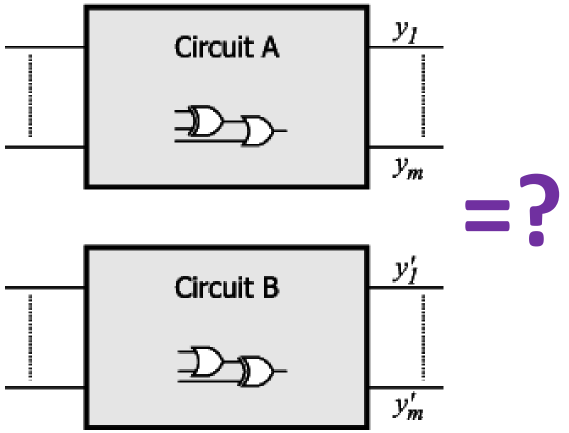
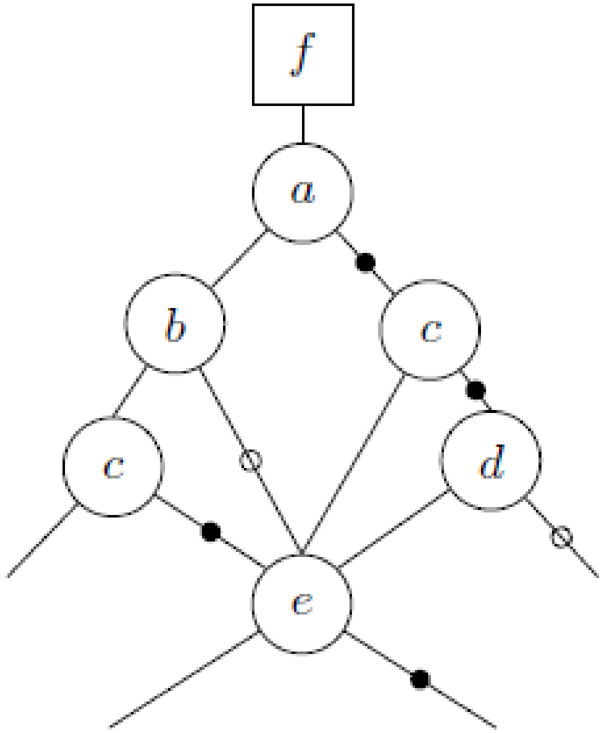
Motivation – BDDs

- Advantages:
  - Efficient Manipulation
    - Boolean Operations
  - Often small representation
  - Canonical (unique) representation
    - If two formulas are equivalent, then their BDD representations are equivalent

Application:
Circuit Equivalence Checking





Outline

- What are Binary Decision Diagrams (BDDs)?
  - Intuitive Explanation
  - Formal Definition
  - Reduced-Ordered BDDs
    - for us, a BDD is always reduced and ordered

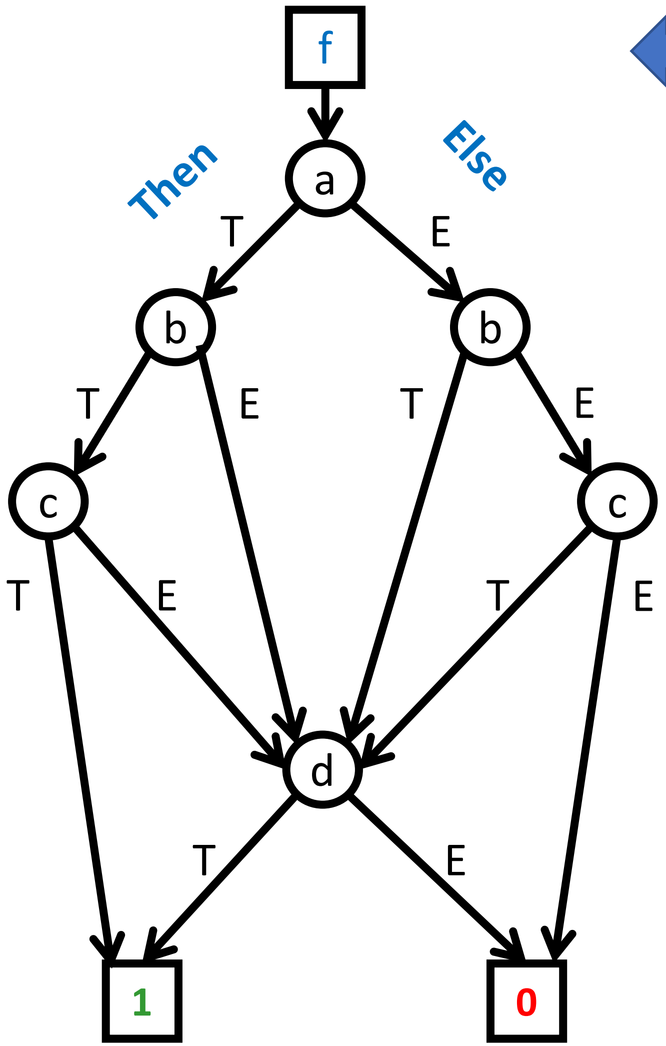
- **Represent a formula in propositional logic as BDD**
- **From the BDD, derive the formula that is represented by a BDD**

Learning Outcomes

After this lecture…

1. students can define and explain BDDs.
   - BDD = reduced and ordered binary decision diagram
   - Define and explain its elements and their meaning
2. students can represent a formula in propositional logic as BDD.
3. students can derive the formula that is represented by a BDD.
4. students can state properties of BDDs.
   - advantages, disadvantages

Binary Decision Diagram (BDD) → Initial Representation

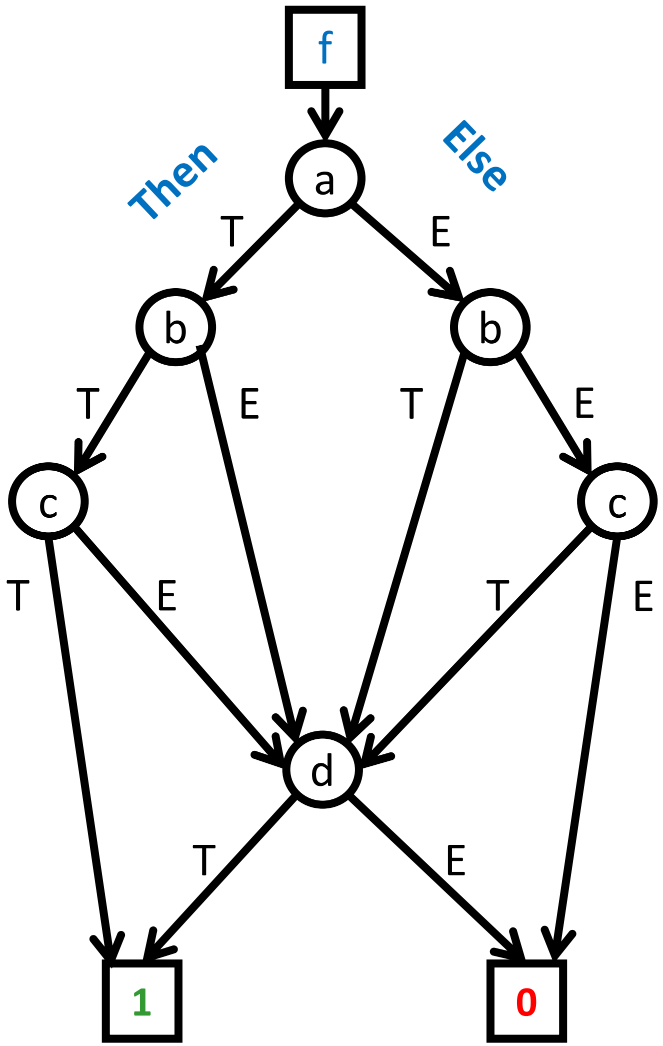


Function node:
Represents formula f

Internal nodes:
Variables of f

Terminal nodes:
Represent truth values

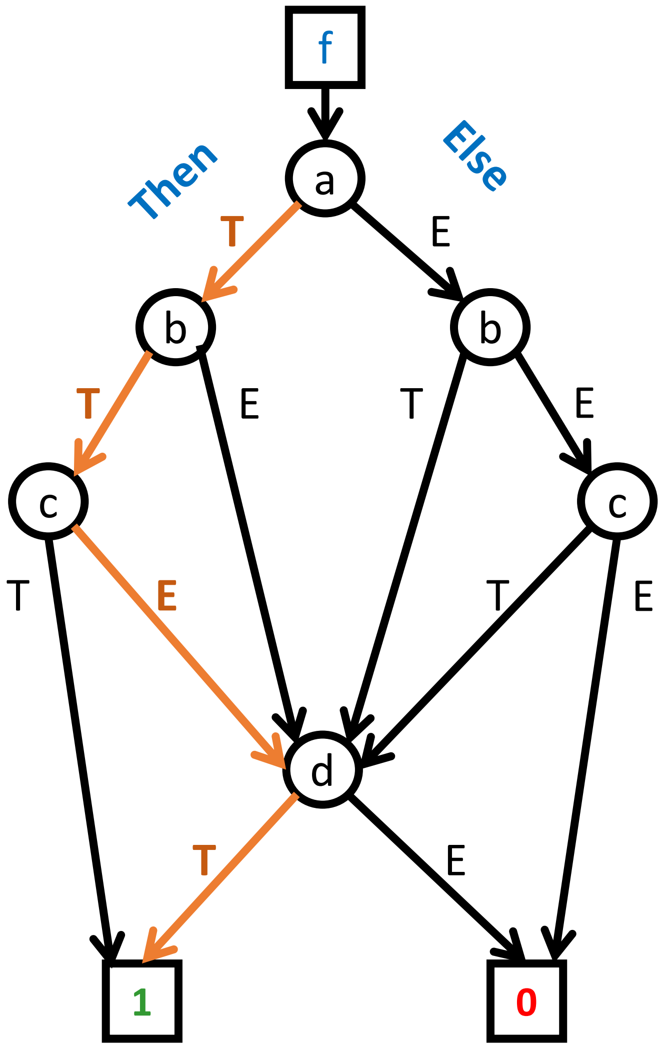
Binary Decision Diagram (BDD)



Given $M := \{a = T, b = T, c = T, d = T\}$

Does it hold that $M \vDash f$?

Binary Decision Diagram (BDD)

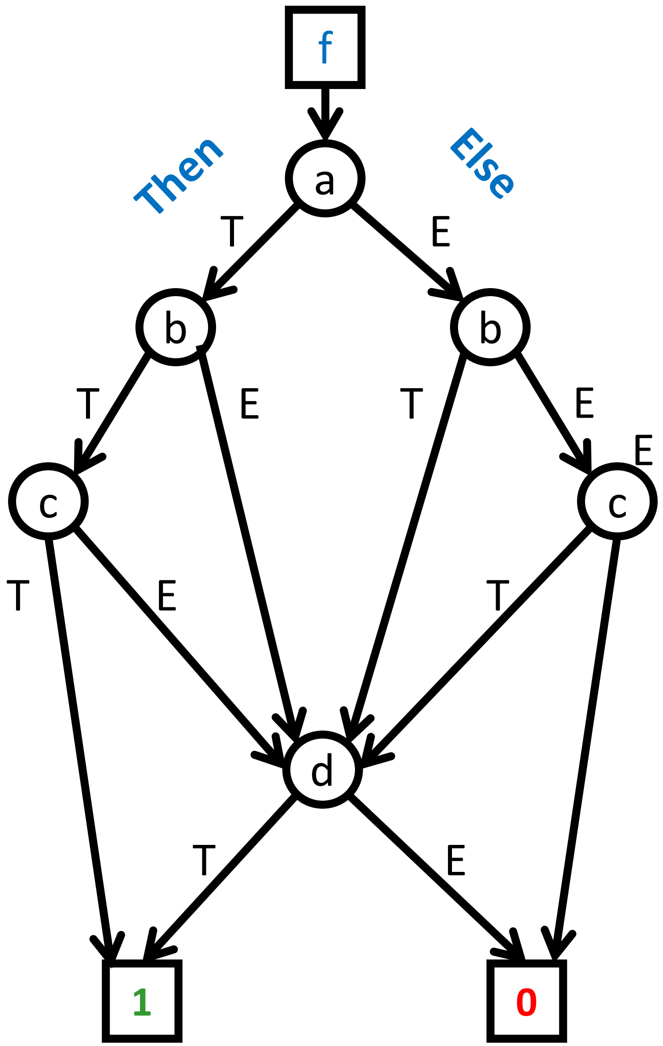


Given $M := \{a = T, b = T, c = T, d = T\}$

Does it hold that M $\vDash f$?

***M* is a satisfying assignment**

Binary Decision Diagram (BDD)

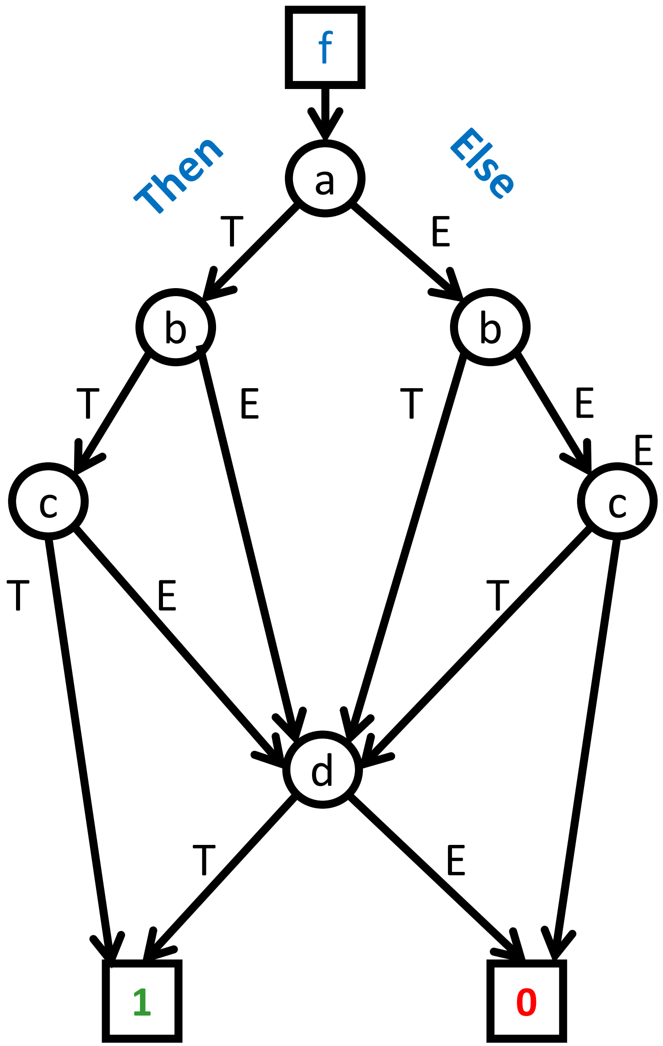


$M \vDash f$ iff its path in BDDs ends in terminal node **1**

$M \nvDash f$ iff its path in BDDs ends in terminal node **0**

How can we find the formula f that is represented by this BDD?

Binary Decision Diagram (BDD)



$M \models f$ iff its path in BDDs ends in terminal node 1

$M \not\models f$ iff its path in BDDs ends in terminal node 0

How can we find the formula f that is represented by this BDD?

Represent f in DNF by enumerating all paths (models) that end in 1

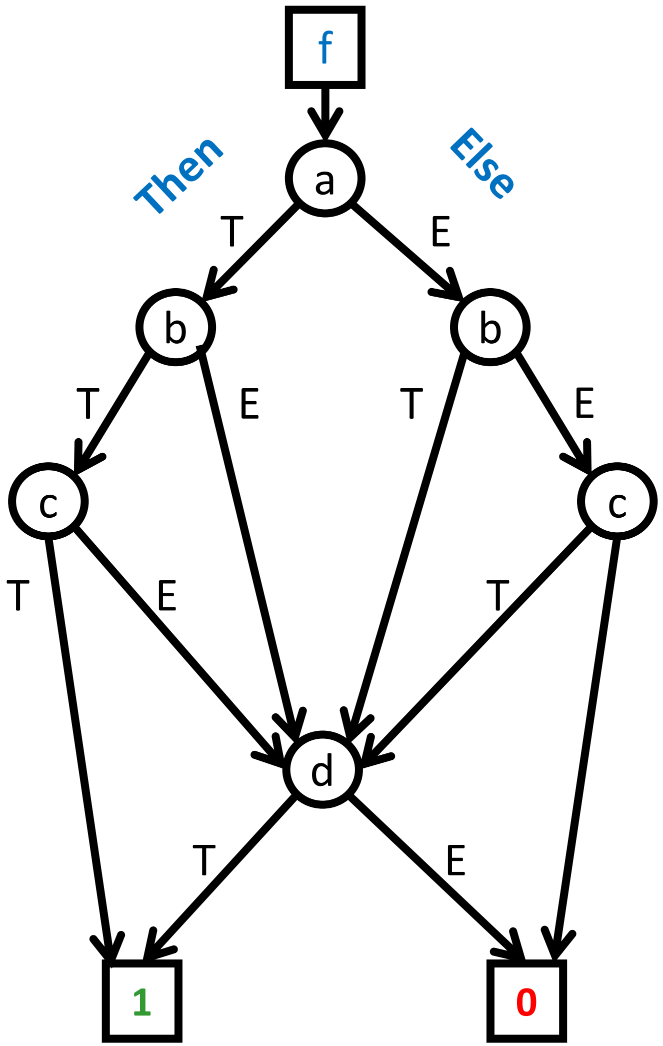
Or

Exclude all paths that end in 0

Binary Decision Diagram (BDD)

f is in disjunctive normal form (DNF) if it is a **disjunction** of **conjunctions**.

$$f = (a \wedge b) \vee (\neg a \vee c)$$

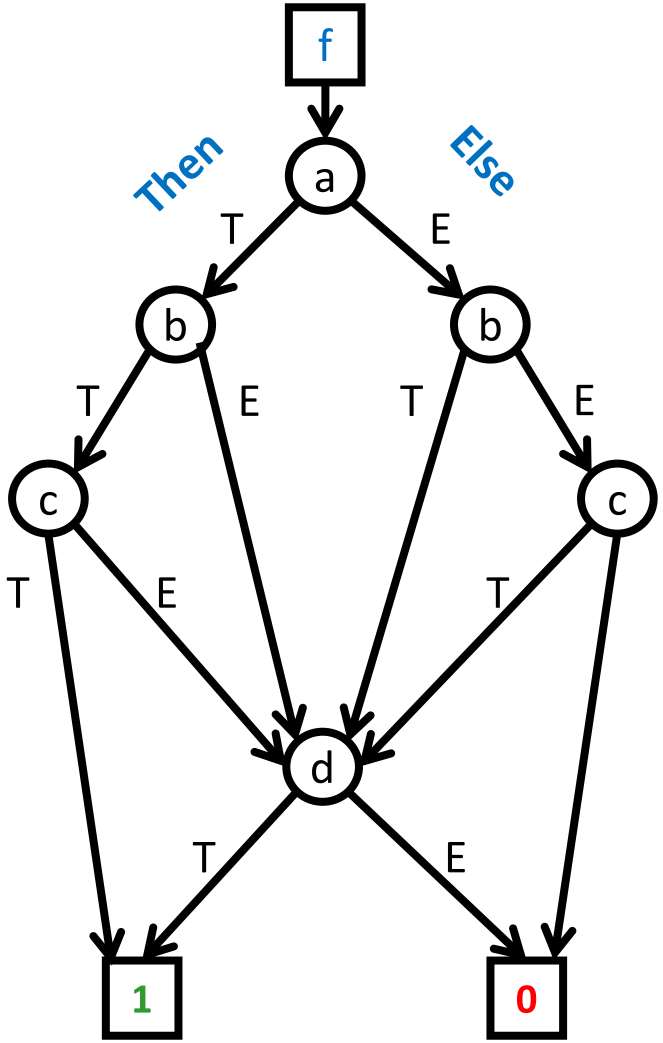


Represent f in DNF by enumerating all paths (models) that end in 1

Binary Decision Diagram (BDD)

f is in disjunctive normal form (DNF) if it is a **disjunction** of **conjunctions**.

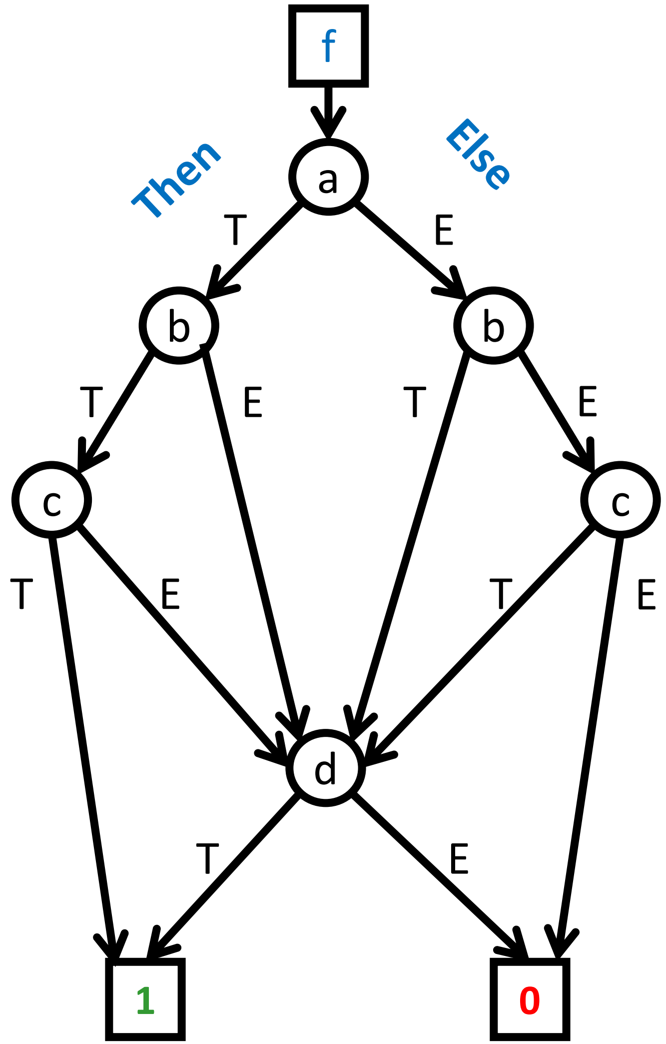
$$f = (a \land b) \lor (\neg a \lor c)$$

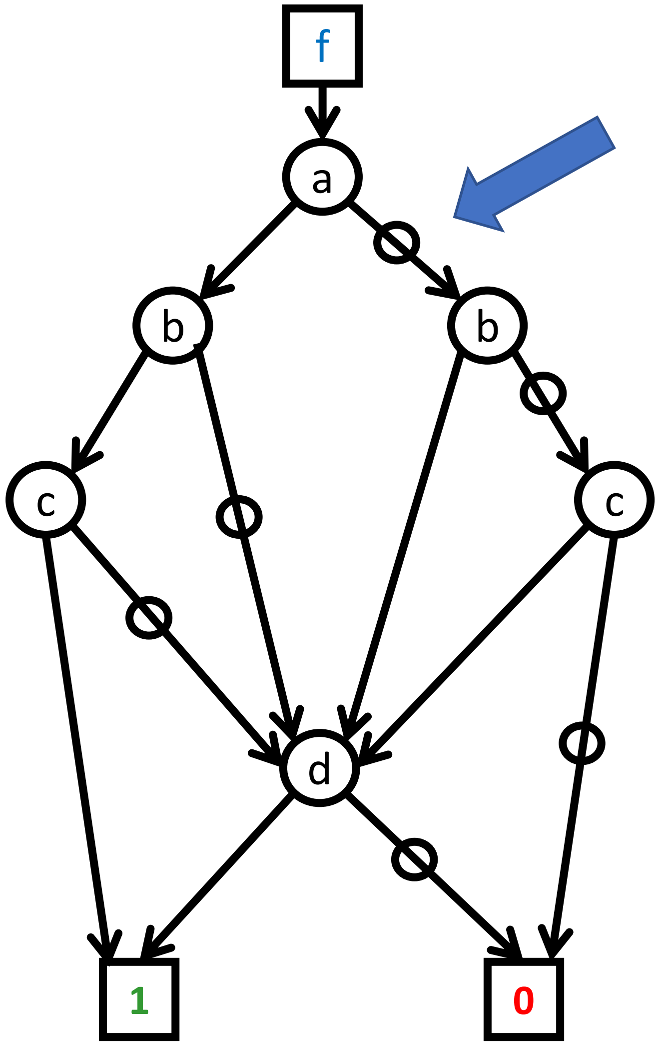


Represent f in DNF by enumerating all paths (models) that end in 1

$$f := (a \land b \land c) \lor (a \land b \land \neg c \land d) \lor (a \land \neg b \land d) \lor (\neg a \land b \land d) \lor (\neg a \land \neg b \land \mathrm{c} \land d)$$

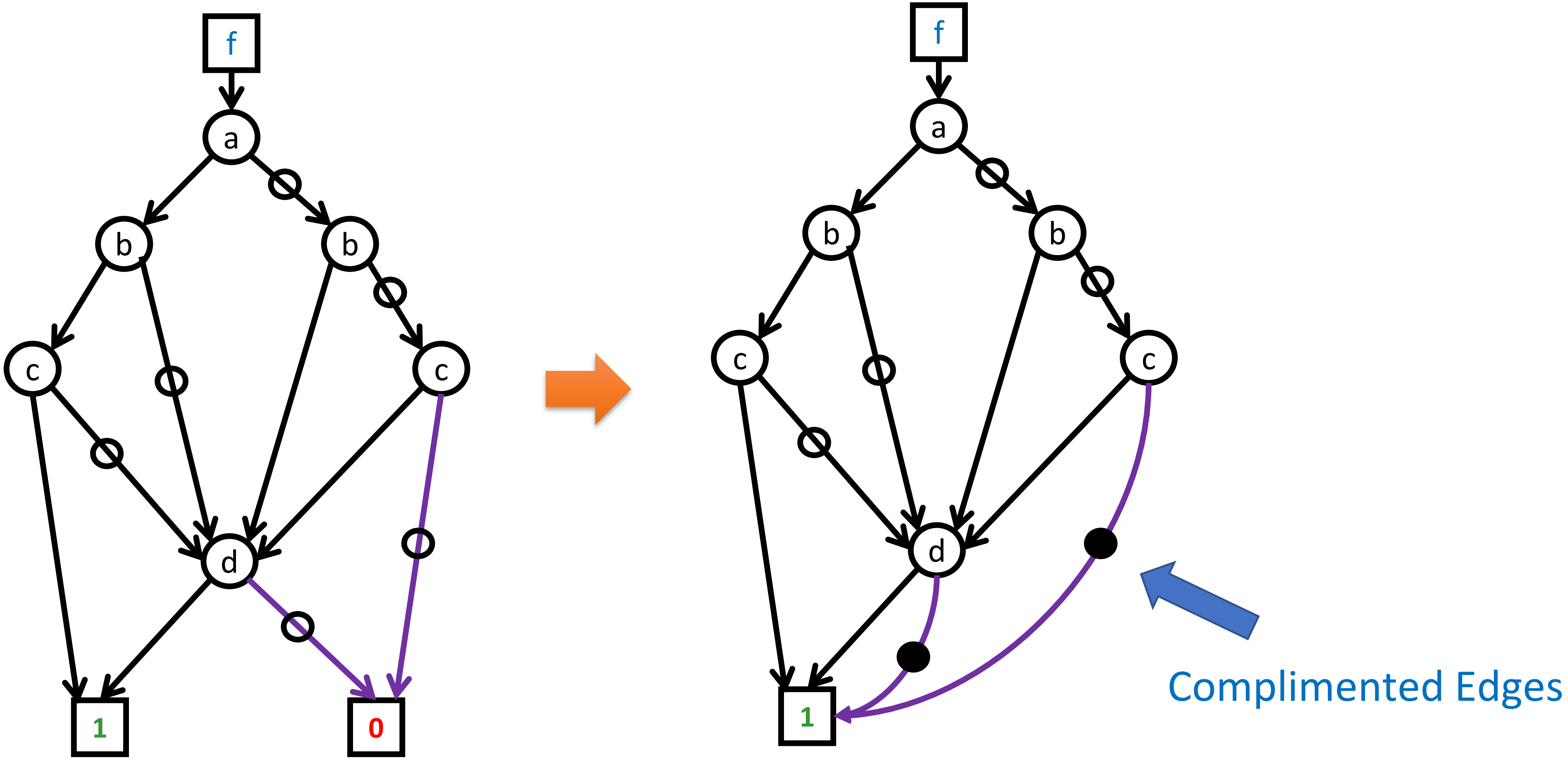
BDD - Representation





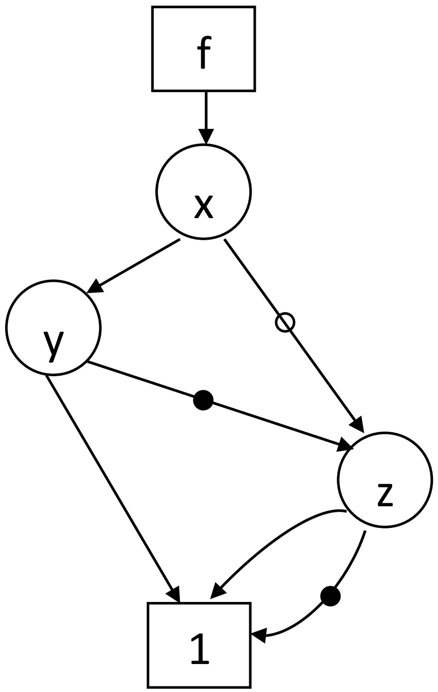
Else-edges are marked by circles

BDD - Representation



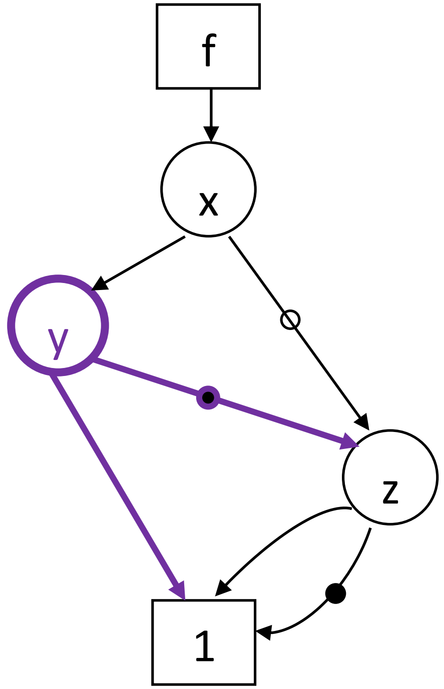
Definition of BDDs

- Directed Acyclic Graph
- $(V \cup f \cup \{\mathbf{1}\},\ E)$
  - Internal Nodes $v \in V$
  - Function Node f
    - Represents propositional formula f
    - May have additional nodes for subformulas
  - Terminal Node $\mathbf{1}$
    - Represents the truth value T
  - Edges E
    - “Complement” attribute



Definition of BDDs: Internal Node

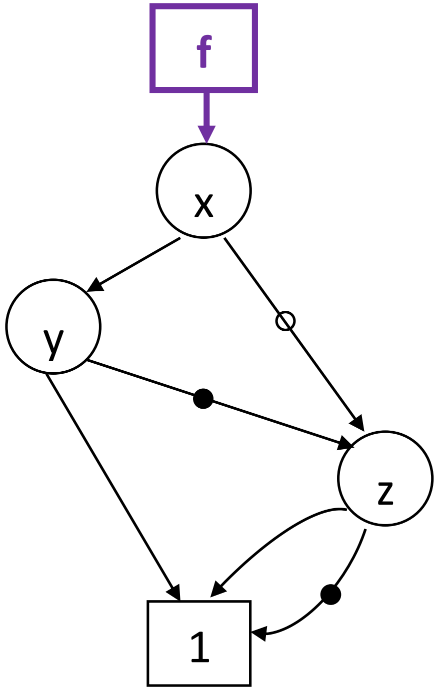
- Label $l(v) \in \{x_1, \dots, x_n\}$
  - Variables of f
- Out–degree: 2
  - Then-Edge T
  - Else-Edge E
    - Marked with (empty) circle
    - Can have complement attribute (full cycle)



Definition of BDDs: Function Node

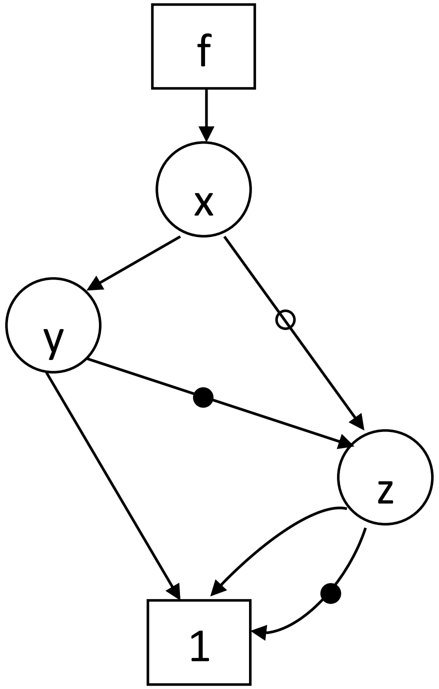
- Represents Boolean Formula f

- In-degree: 0
- Out-degree: 1
  - Edge can have complement attribute



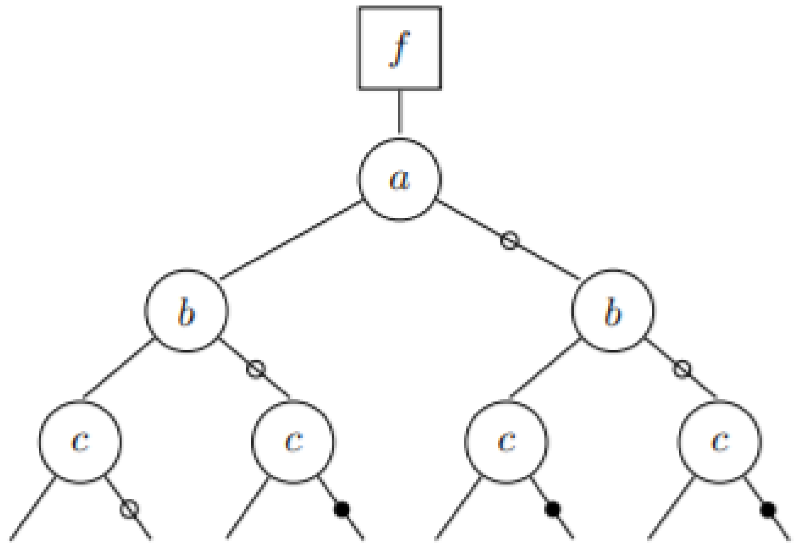
Definition of BDDs: Terminal Node

- Constant Function **True**
- Out-degree: 0

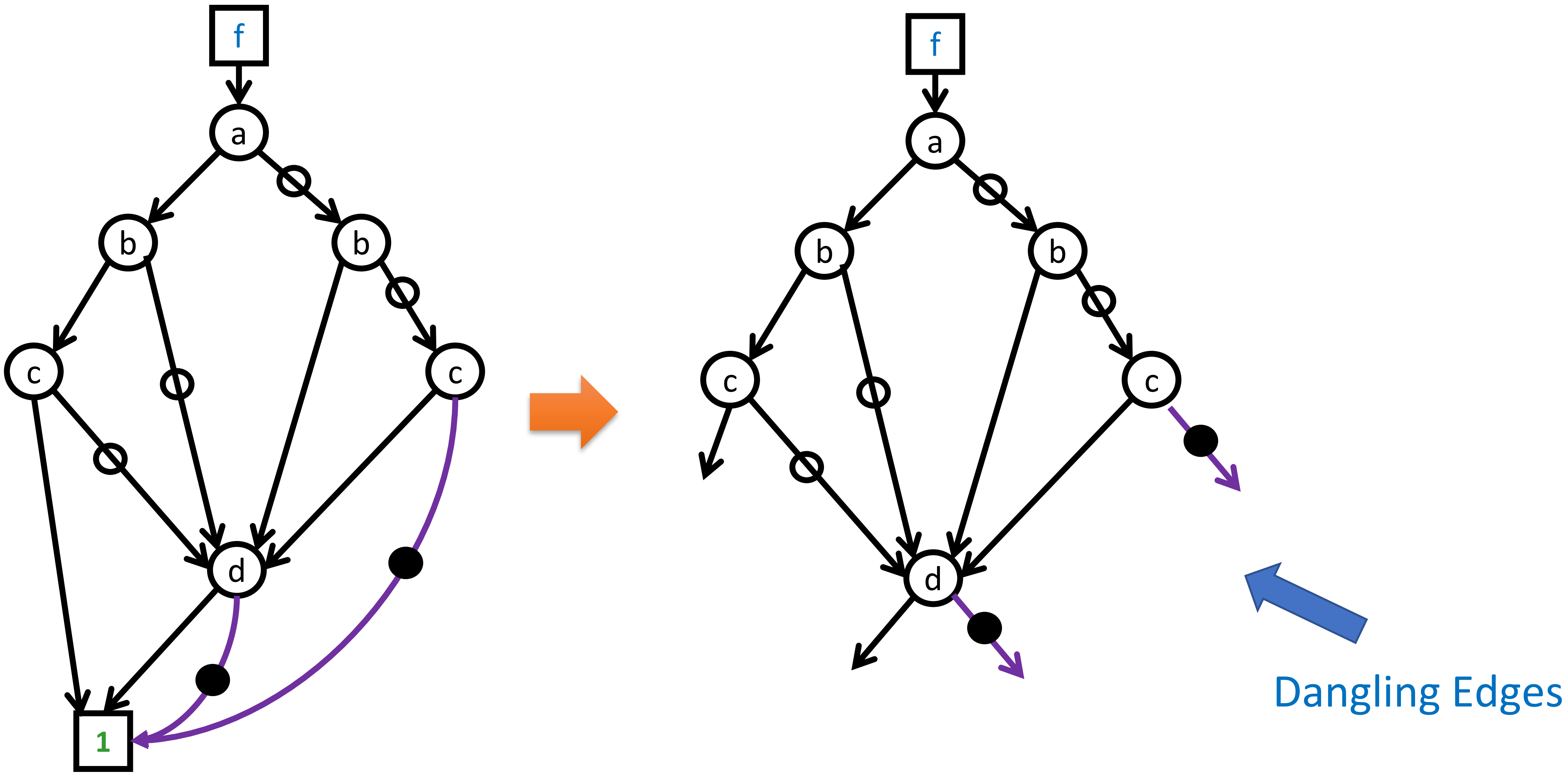


Size of a BDD

- **Worst case: exponential**
  - If each internal node has 2 Sub-BDDs, then BDD has $2^n - 1$ internal nodes
- Often: BDDs contain much redundancy
  - Obtain compact BDD by removing redundancies

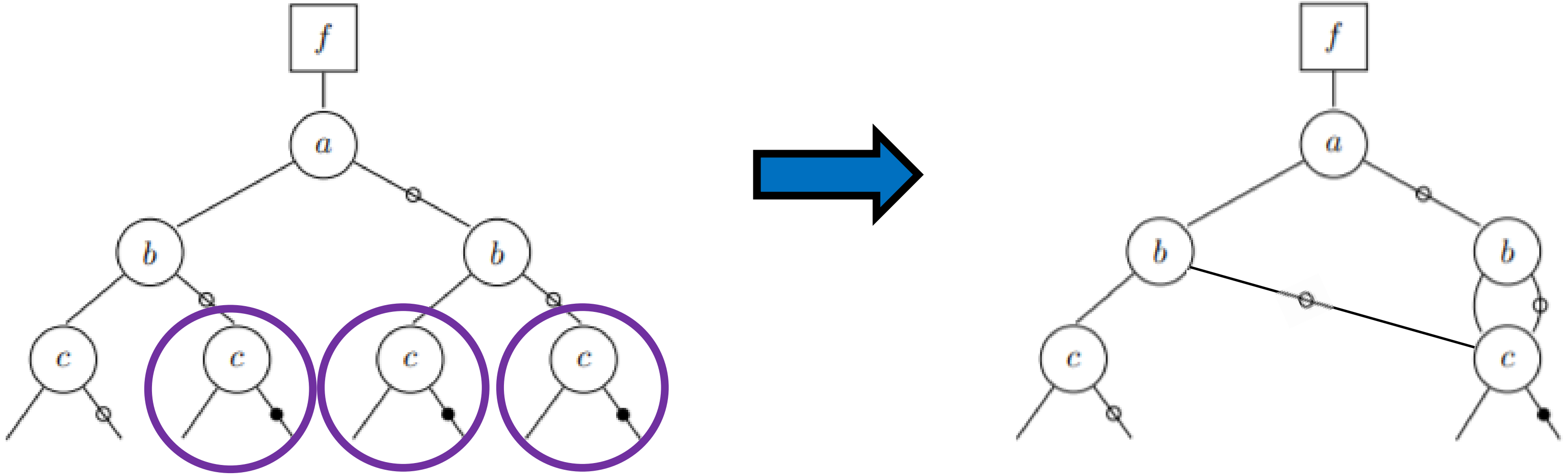


BDD – Representation



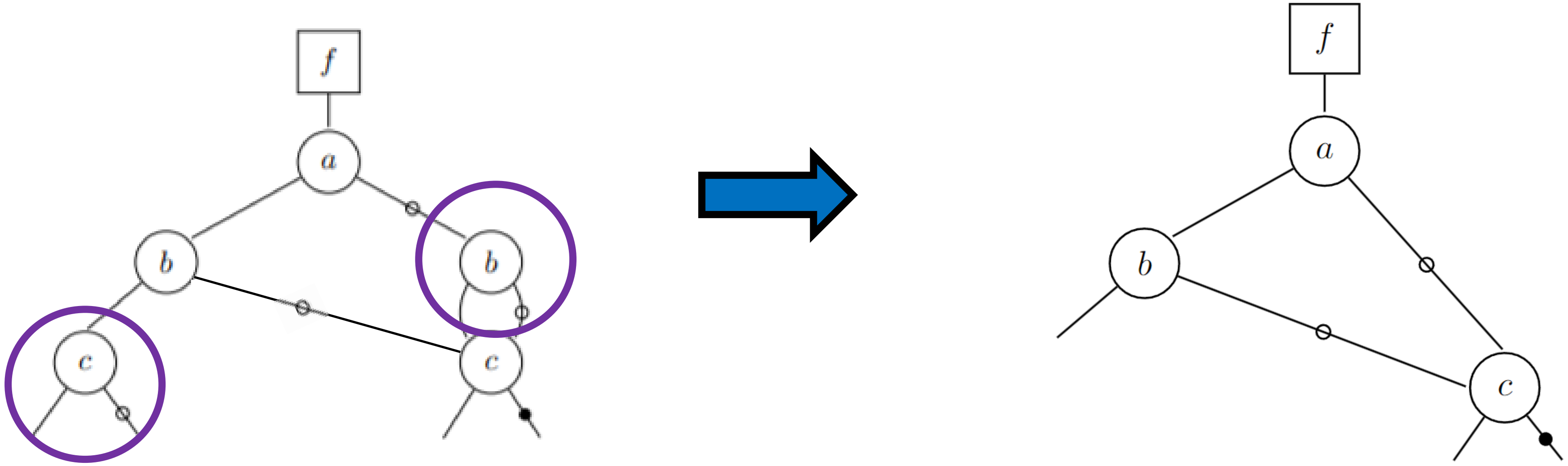
Reduced Ordered BDD

1. No duplicate sub-BDDs



Reduced Ordered BDD

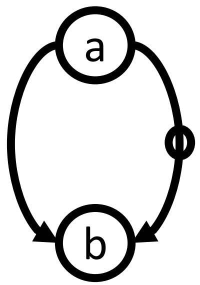
1. No duplicate sub-BDDs
2. No redundant nodes



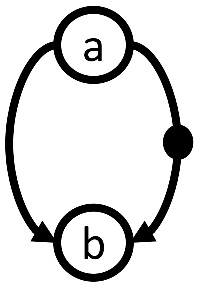
Reduced Ordered BDD

1. No duplicate sub-BDDs
2. No redundant nodes

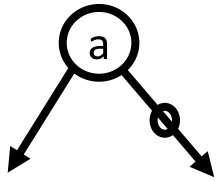
Redundant



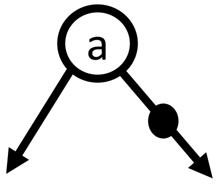
NOT Redundant



Redundant
(special case)



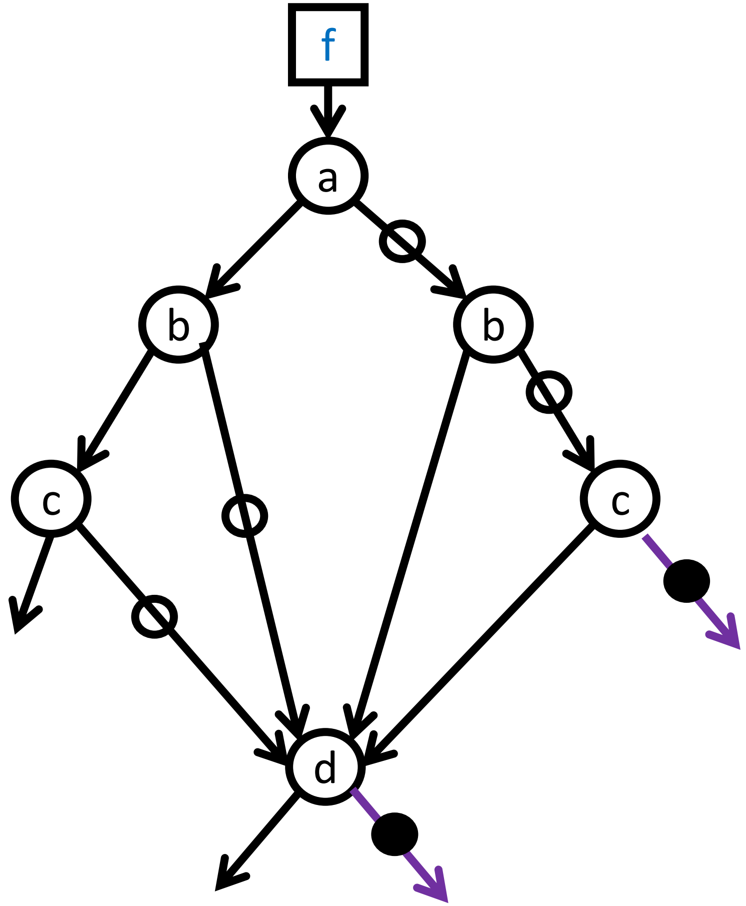
NOT Redundant
(special case)



Reduced Ordered BDD

1. No duplicate sub-BDDs
2. No redundant nodes

3. Ordering on the variables along any path
   - E.g., $a < b < c < \mathrm{d}$



Reduced Ordered BDD

1. No duplicate sub-BDDs
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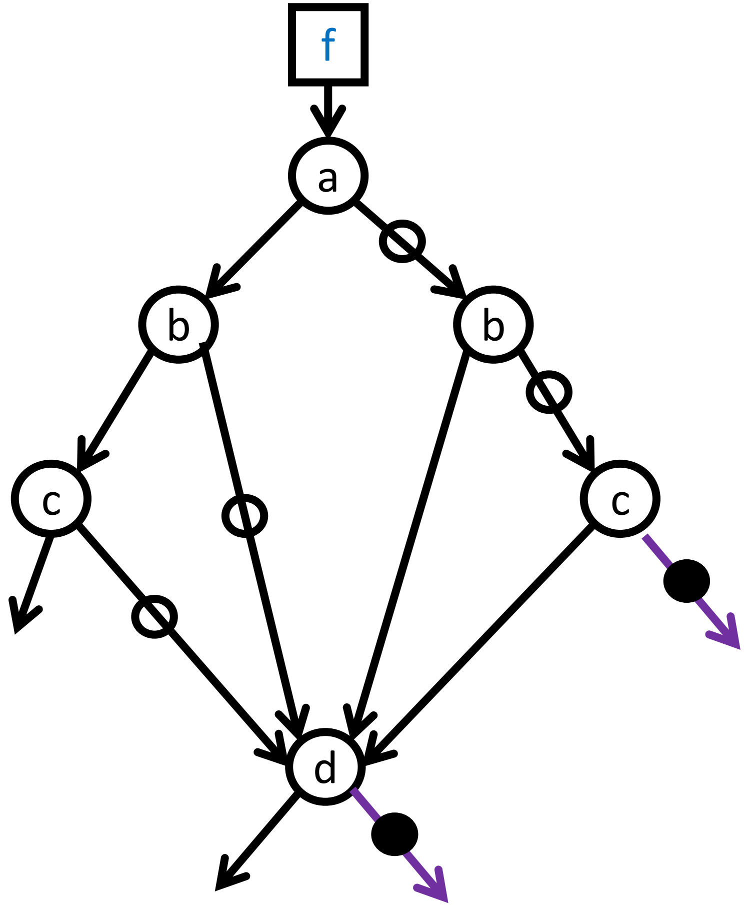
   - E.g., $a < b < c < d$

A reduced and ordered BDD gives a **canonical** representation of a formula

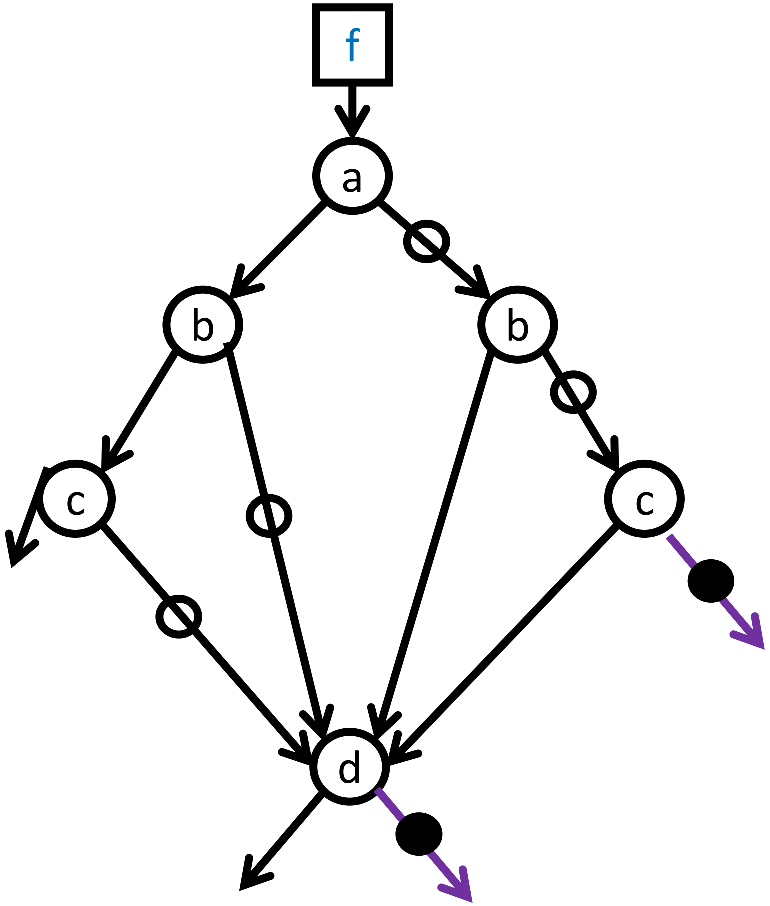


Reduced Ordered BDD

A reduced and ordered BDD gives a **canonical** representation of a formula

How can we use canonicity to decide validity and satisfiability in constant time?

Assume: BDD is given

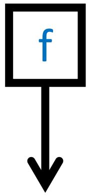


Reduced Ordered BDD

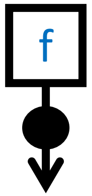
How can we use canonicity to decide validity and satisfiability in constant time?

Assume: BDD is given

BDD of a valid formula



BDD of UNSAT formula



Formula f is SAT if BDD looks different than this

From now on

BDDs are always reduced and ordered!

Outline

- What are Binary Decision Diagrams (BDDs)?
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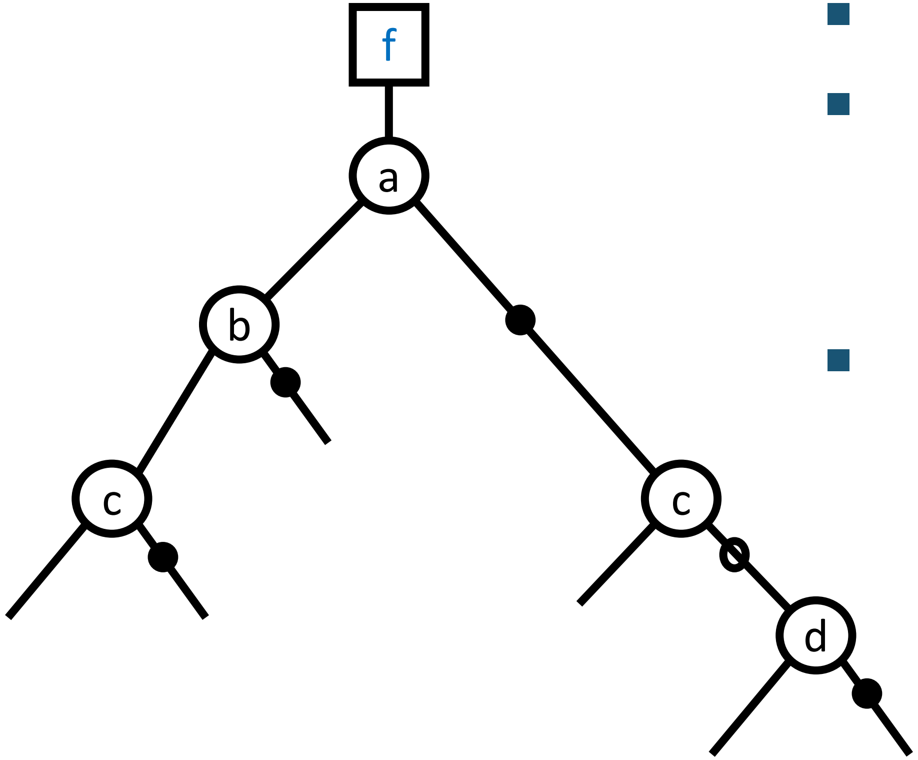
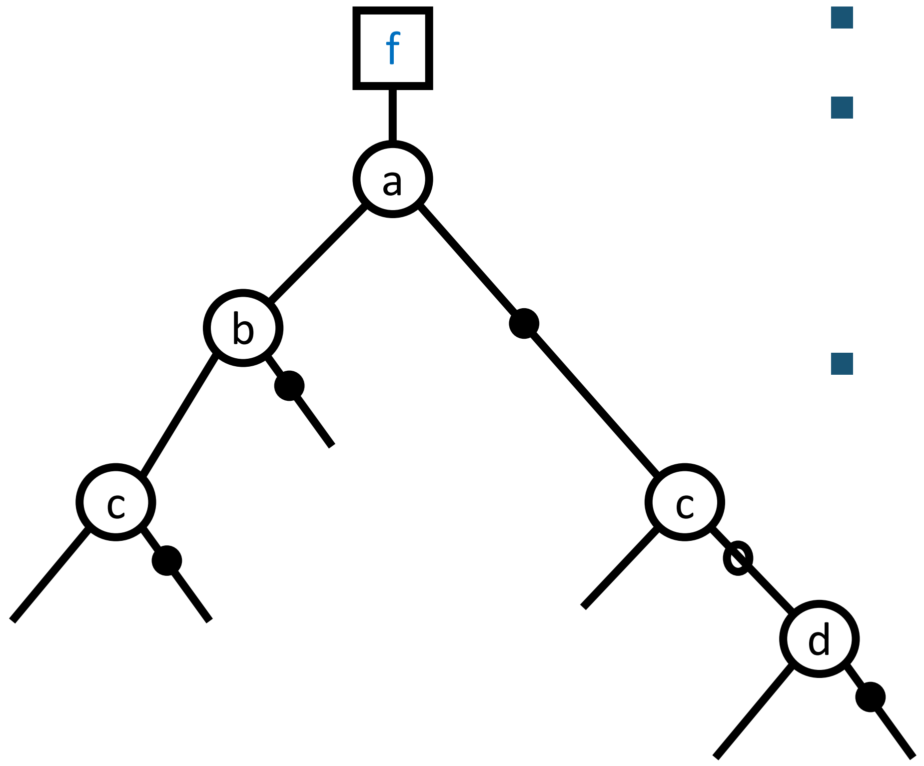
- **Represent a formula in propositional logic as BDD**
- **From the BDD, derive the formula that is represented by a BDD**

From BDD to Formula



- Complement flips truth value
- Satisfying model
  - Represented by path with **even number of negations**
- Build DNF from satisfying models

From BDD to Formula



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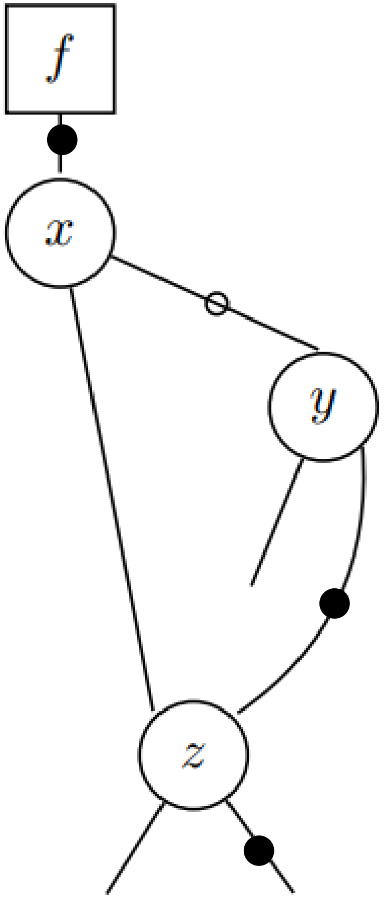
$$f = (a \land b \land c) \lor (\neg a \land \neg c \land \neg d)$$

From BDD to Formula



- Complement flips truth value
- Satisfying model
  - Represented by path with **even number of negations**
- Build DNF from satisfying models

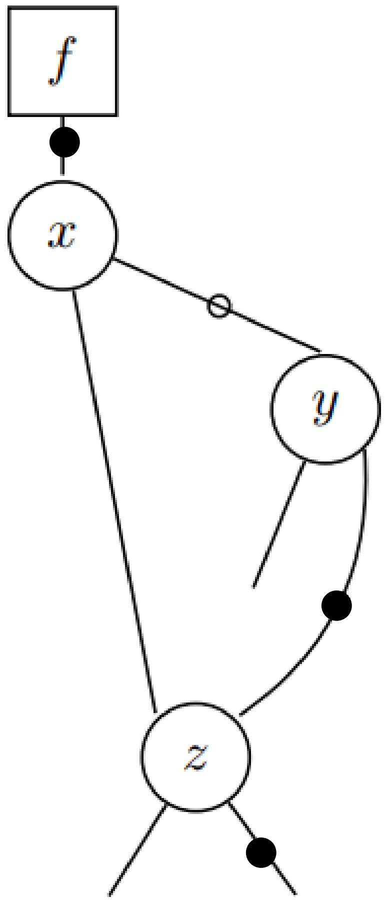
What is the formula f represented by this BDD?

(a) $f = (x \wedge z) \vee (\neg x \wedge y) \vee (\neg x \wedge \neg y \wedge \neg z)$

(b) $f = (x \wedge \neg z) \vee (\neg x \wedge \neg y \wedge z) \vee (x \wedge z)$

(c) $f = (x \wedge \neg z) \vee (\neg x \wedge \neg y \wedge z)$

From BDD to Formula



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- Build DNF from satisfying models

(a) $f = (x \wedge z) \vee (\neg x \wedge y) \vee (\neg x \wedge \neg y \wedge \neg z)$

(b) $f = (x \wedge \neg z) \vee (\neg x \wedge \neg y \wedge z) \vee (x \wedge z)$

(c) $\boldsymbol{f = (x \wedge \neg z) \vee (\neg x \wedge \neg y \wedge z)}$

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- What are Binary Decision Diagrams (BDDs)?
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- **Represent a formula in propositional logic as BDD**
- **From the BDD, derive the formula that is represented by a BDD**

From Formula to BDD

1. Compute all Cofactors
2. Draw ROBDD from Cofactors
3. Shift Negations Upwards

From Formula to BDD – Step 1: Cofactors

- Boolean formula f w.r.t. a variable x
  - Positive Cofactor f_x: f with x set to $\top$
  - Negative Cofactor $f_{\neg x}$: f with x set to $\bot$

- Example:
  - $f = (x \wedge y) \vee (\neg x \wedge z)$

$$f_x =$$
$$f_{\neg x} =$$

From Formula to BDD – Step 1: Cofactors

- Boolean formula f w.r.t. a variable x
  - Positive Cofactor f_x: f with x set to $\top$
  - Negative Cofactor $f_{\neg x}$: f with x set to $\bot$

- Example:
  - $f = (x \wedge y) \vee (\neg x \wedge z)$
    - $f_x = y$
    - $f_{\neg x} = z$

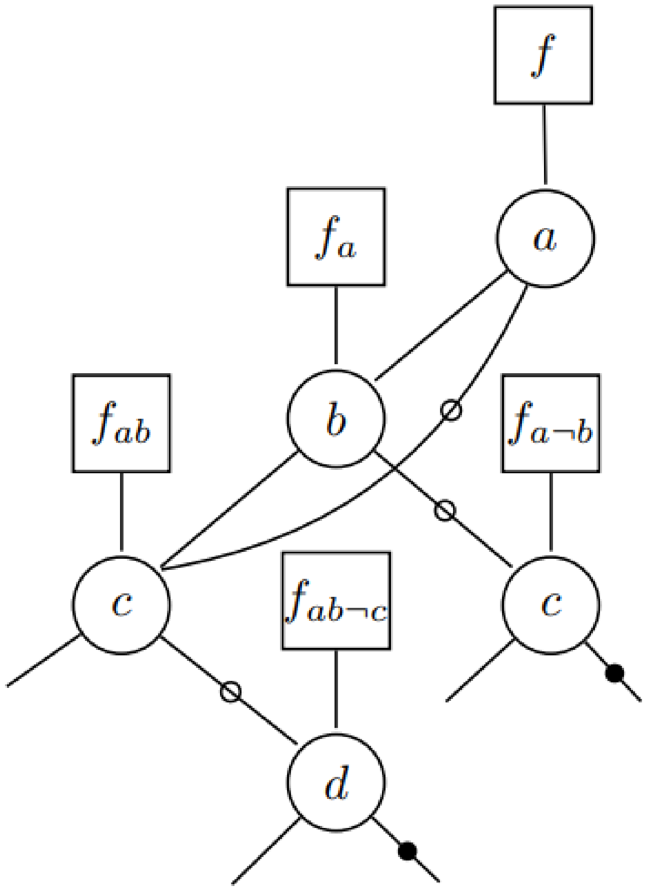
From Formula to BDD

Construct the BDD for the formula $\boldsymbol{f} = ((\boldsymbol{a} \wedge \boldsymbol{b} \vee \neg \boldsymbol{a}) \wedge \neg \boldsymbol{c} \wedge \boldsymbol{d}) \vee \boldsymbol{c}$.
Use the variable order $a < b < c < d$

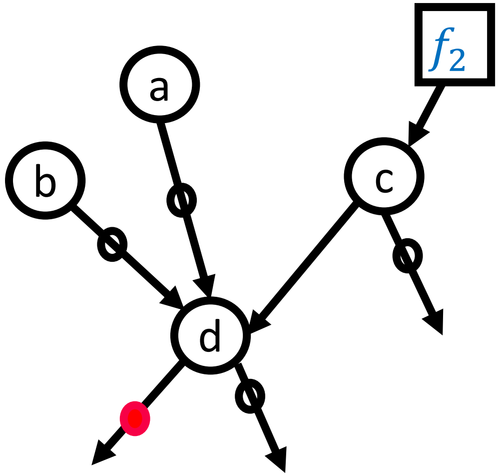
From Formula to BDD

Construct the BDD for the formula $\boldsymbol{f} = ((\boldsymbol{a} \wedge \boldsymbol{b} \vee \neg \boldsymbol{a}) \wedge \neg \boldsymbol{c} \wedge \boldsymbol{d}) \vee \boldsymbol{c}$.
Use the variable order $a < b < c < d$

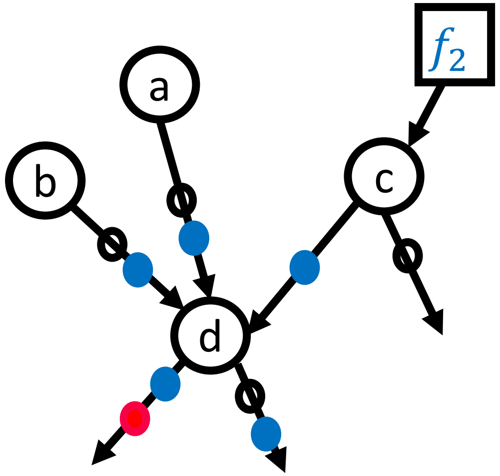
$$
\begin{aligned}
f_a &= b \wedge \neg c \wedge d \vee c \\
f_{ab} &= \neg c \wedge d \vee c \\
f_{abc} &= \top \\
f_{ab\neg c} &= d \\
f_{ab\neg cd} &= \top \\
f_{ab\neg c\neg d} &= \bot \\
f_{a\neg b} &= c \\
f_{a\neg bc} &= \top \\
f_{a\neg b\neg c} &= \bot \\
f_{\neg a} &= \neg c \wedge d \vee c = f_{ab}
\end{aligned}
$$



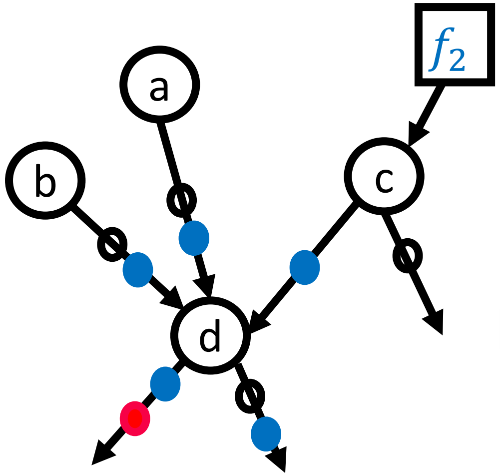
From Formula to BDD – Step 3: Shift Negations Upwards

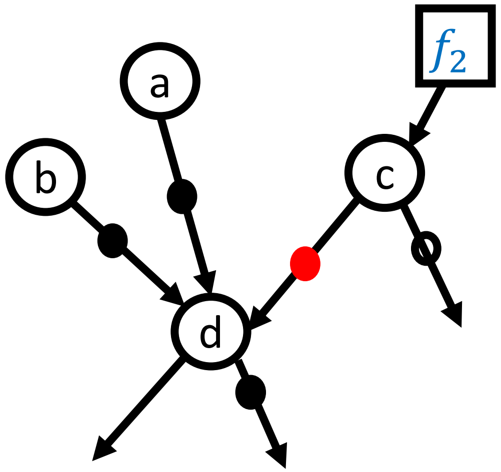


From Formula to BDD – Step 3: Shift Negations Upwards

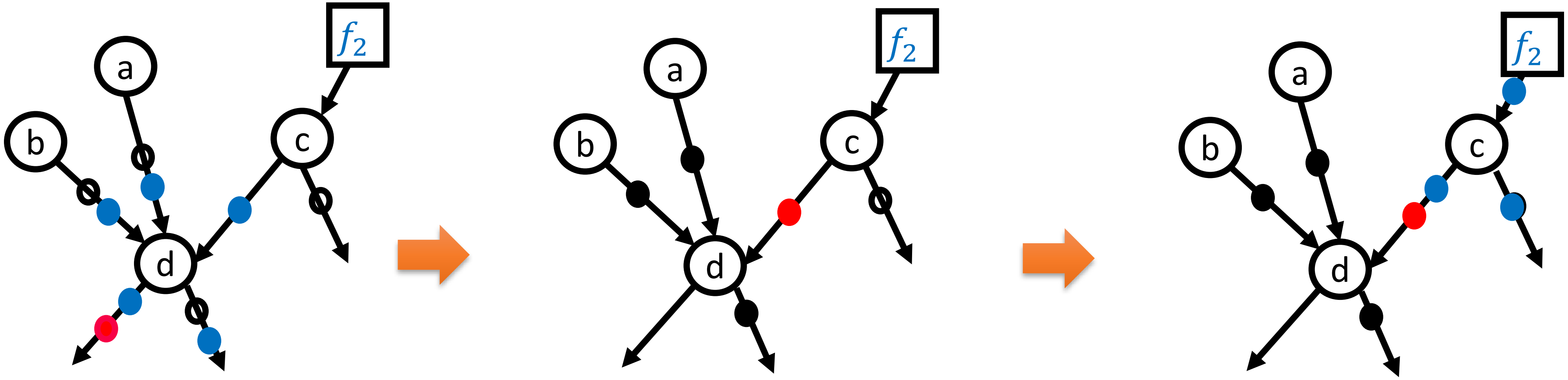


From Formula to BDD – Step 3: Shift Negations Upwards

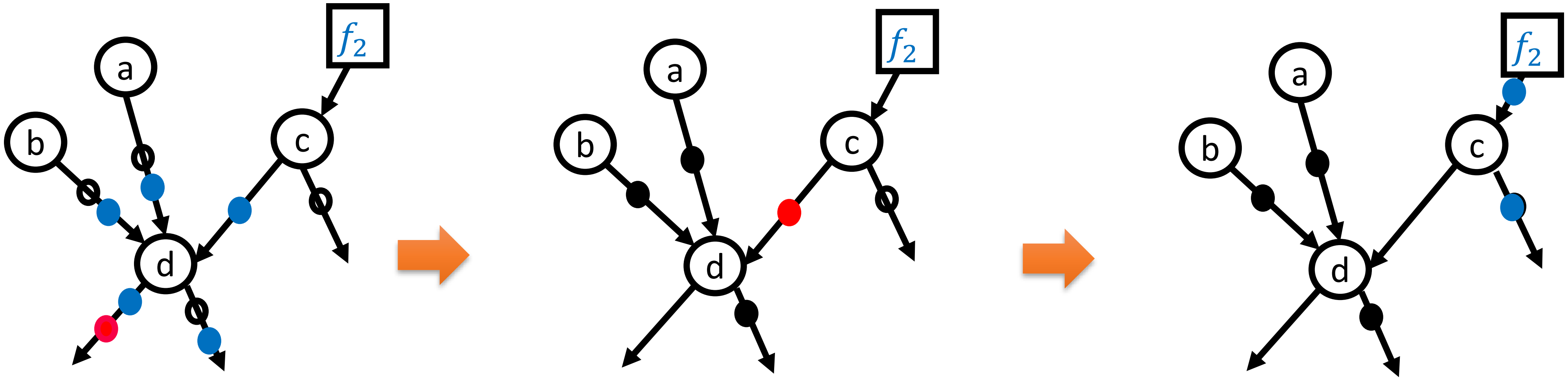




From Formula to BDD – Step 3: Shift Negations Upwards



From Formula to BDD – Step 3: Shift Negations Upwards



From Formula to BDD

Construct the BDD for the formula $\boldsymbol{f} = (\boldsymbol{a} \land \neg \boldsymbol{c}) \lor (\neg \boldsymbol{a} \land (\boldsymbol{b} \lor (\neg \boldsymbol{b} \land \boldsymbol{c})))$.
Use the variable order $a < b < c < d$

From Formula to BDD

Construct the BDD for the formula $\boldsymbol{f} = (\boldsymbol{a} \land \neg \boldsymbol{c}) \lor (\neg \boldsymbol{a} \land (\boldsymbol{b} \lor (\neg \boldsymbol{b} \land \boldsymbol{c})))$.
Use the variable order $a < b < c < d$

$f_a = \neg c$

$f_{ac} = \bot$

$f_{a\neg c} = \top$

$f_{\neg a} = b \lor (\neg b \land c)$

$f_{\neg ab} = \top$

$f_{\neg a \neg b} = c = \neg f_a$

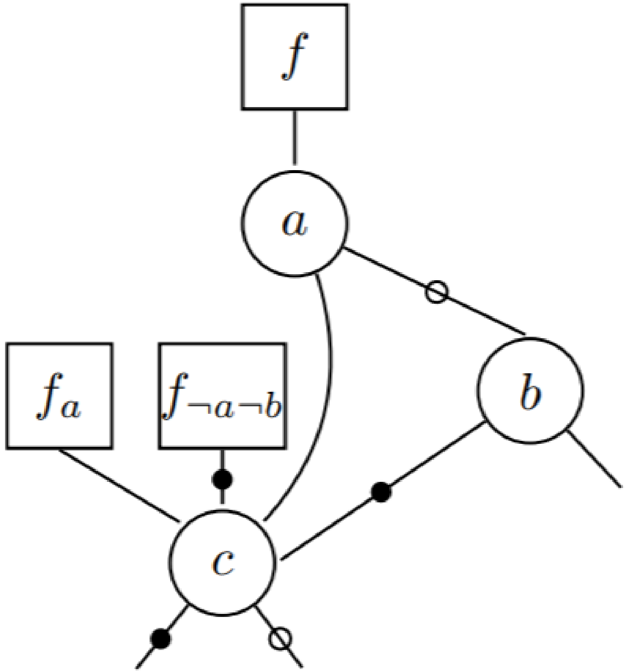
From Formula to BDD

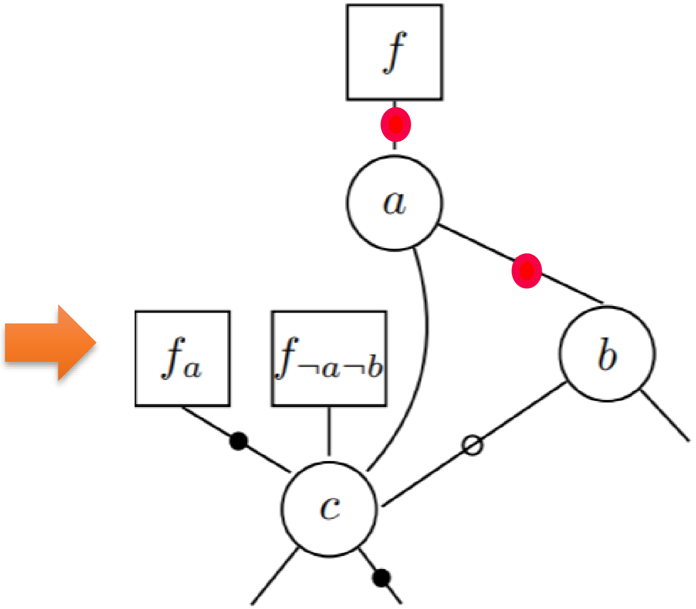
Construct the BDD for the formula $\boldsymbol{f} = (\boldsymbol{a} \wedge \neg \boldsymbol{c}) \vee (\neg \boldsymbol{a} \wedge (\boldsymbol{b} \vee (\neg \boldsymbol{b} \wedge \boldsymbol{c})))$.
Use the variable order $a < b < c < d$

Details: Next slide

$f_a = \neg c$
$f_{ac} = \bot$
$f_{a\neg c} = \top$

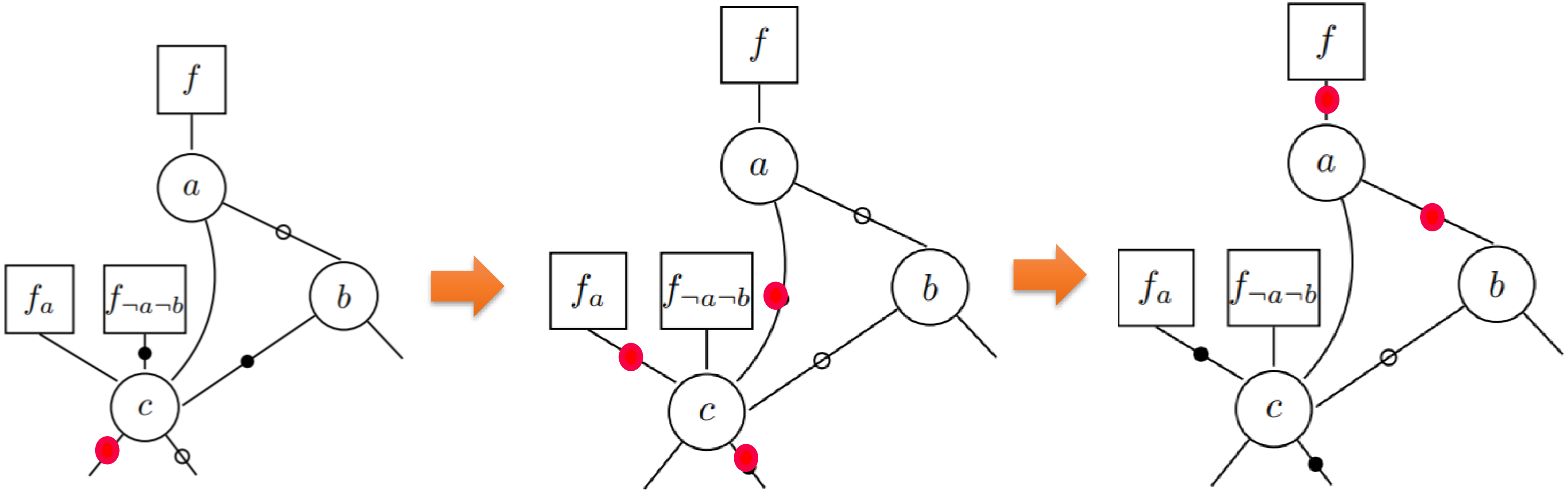
$f_{\neg a} = b \vee (\neg b \wedge c)$
$f_{\neg ab} = \top$
$f_{\neg a \neg b} = c = \neg f_a$





From Formula to BDD

Construct the BDD for the formula $\boldsymbol{f} = (\boldsymbol{a} \land \neg \boldsymbol{c}) \lor (\neg \boldsymbol{a} \land (\boldsymbol{b} \lor (\neg \boldsymbol{b} \land \boldsymbol{c})))$.
Use the variable order $a < b < c < d$



From Formula to BDD

Construct the BDD for the formula

$$\boldsymbol{f = (a \leftrightarrow b) \land (c \leftrightarrow d)}$$

Use the variable order $a < b < c < d$

From Formula to BDD

Construct the BDD for the formula

$$\boldsymbol{f = (a \leftrightarrow b) \wedge (c \leftrightarrow d)}$$

Use the variable order $a < b < c < d$

$f_a = b \wedge (c \leftrightarrow d)$

$f_{ab} = c \leftrightarrow d$

$f_{abc} = d$

$f_{abcd} = \top$

$f_{abc\neg d} = \bot$

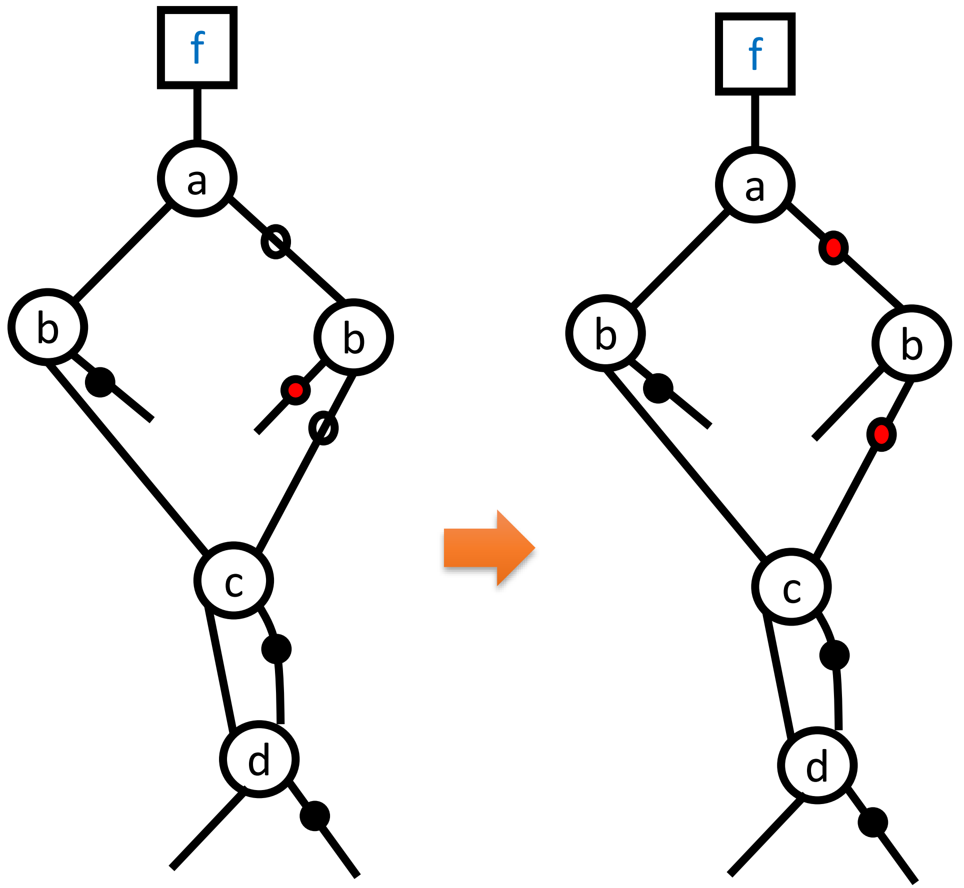
$f_{ab\neg c} = \neg d = \neg f_{abc}$

$f_{a\neg b} = \bot$

$f_{\neg a} = \neg b \wedge (c \leftrightarrow d)$

$f_{\neg ab} = \bot$

$f_{\neg a \neg b} = c \leftrightarrow d = f_{ab}$



From Formula to BDD

Construct the BDD for the formula

$$\boldsymbol{f} = (\boldsymbol{a} \leftrightarrow \boldsymbol{b}) \land (\boldsymbol{c} \leftrightarrow \boldsymbol{d})$$

Use the variable order $a < c < b < d$

From Formula to BDD

Construct the BDD for the formula

$$\boldsymbol{f = (a \leftrightarrow b) \land (c \leftrightarrow d)}$$

Use the variable order $a < c < b < d$

$f_a = b \land (c \leftrightarrow d)$

$f_{ac} = b \land d$

$f_{acb} = d$

$f_{acbd} = \top$

$f_{acb\neg d} = \bot$

$f_{ac\neg b} = \bot$

$f_{a\neg c} = b \land \neg d$

$f_{a\neg cb} = \neg d = \neg f_{acb}$

$f_{a\neg c\neg b} = \bot$

$f_{\neg a} = \neg b \land (c \leftrightarrow d)$

$f_{\neg ac} = \neg b \land d$

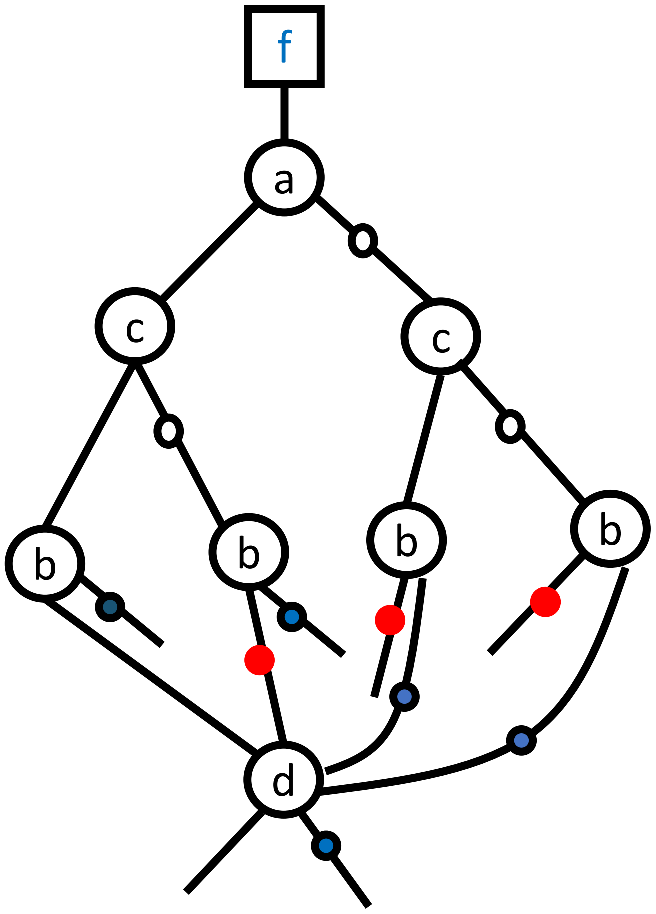
$f_{\neg acb} = \bot$

$f_{\neg ac\neg b} = d = f_{acb}$

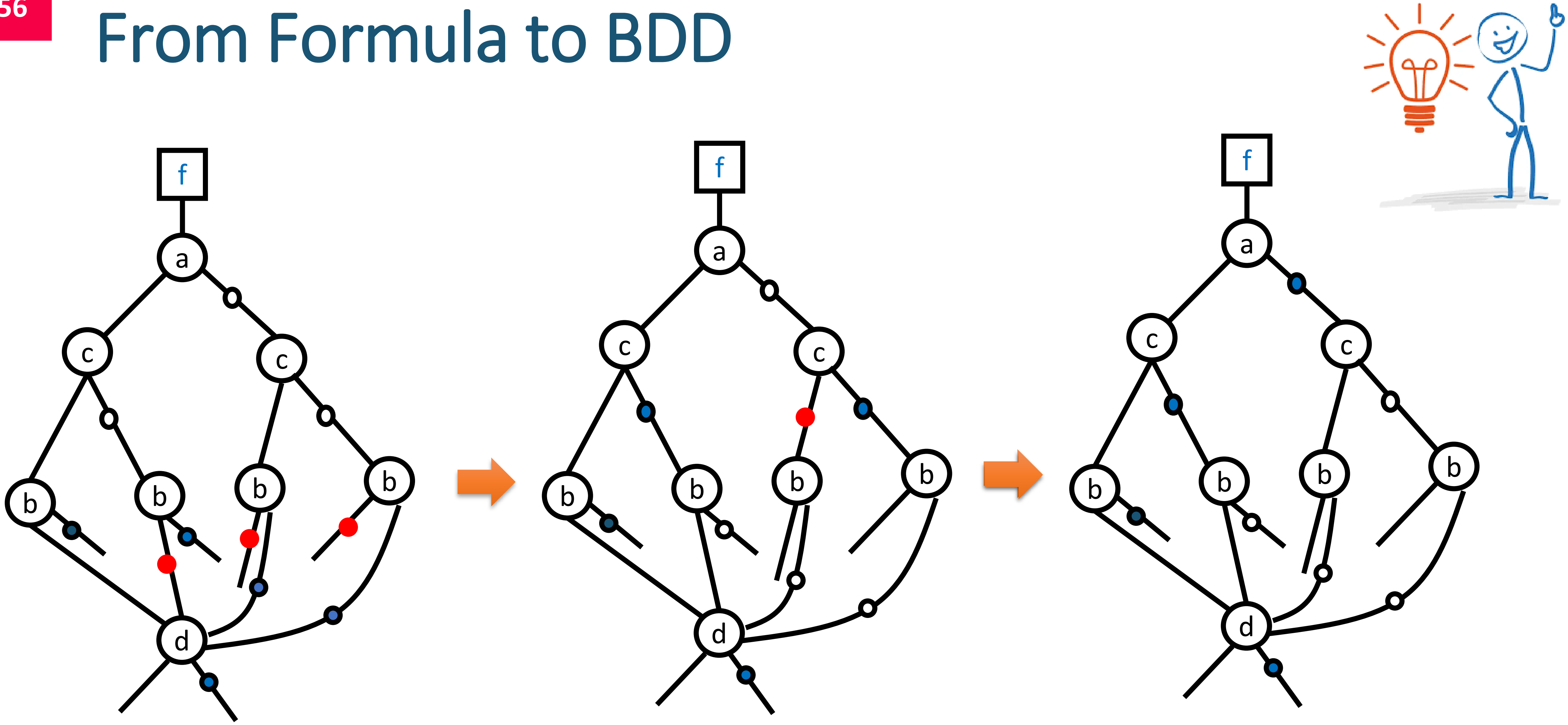
$f_{\neg a\neg c} = \neg b \land \neg d$

$f_{\neg a\neg cb} = \bot$

$f_{\neg a\neg c\neg b} = \neg d = \neg f_{acb}$



From Formula to BDD



Disadvantages of BDDs

Size of BDDs strongly depends on variable order

- Hard to optimize
- Problem to find the optimal variable order is NP complete

Example before: $f = (x_1 \leftrightarrow x_1{}') \wedge (x_2 \leftrightarrow x_2{}') \wedge (x_3 \leftrightarrow x_3{}') \wedge (x_4 \leftrightarrow x_4{}') \wedge \cdots$

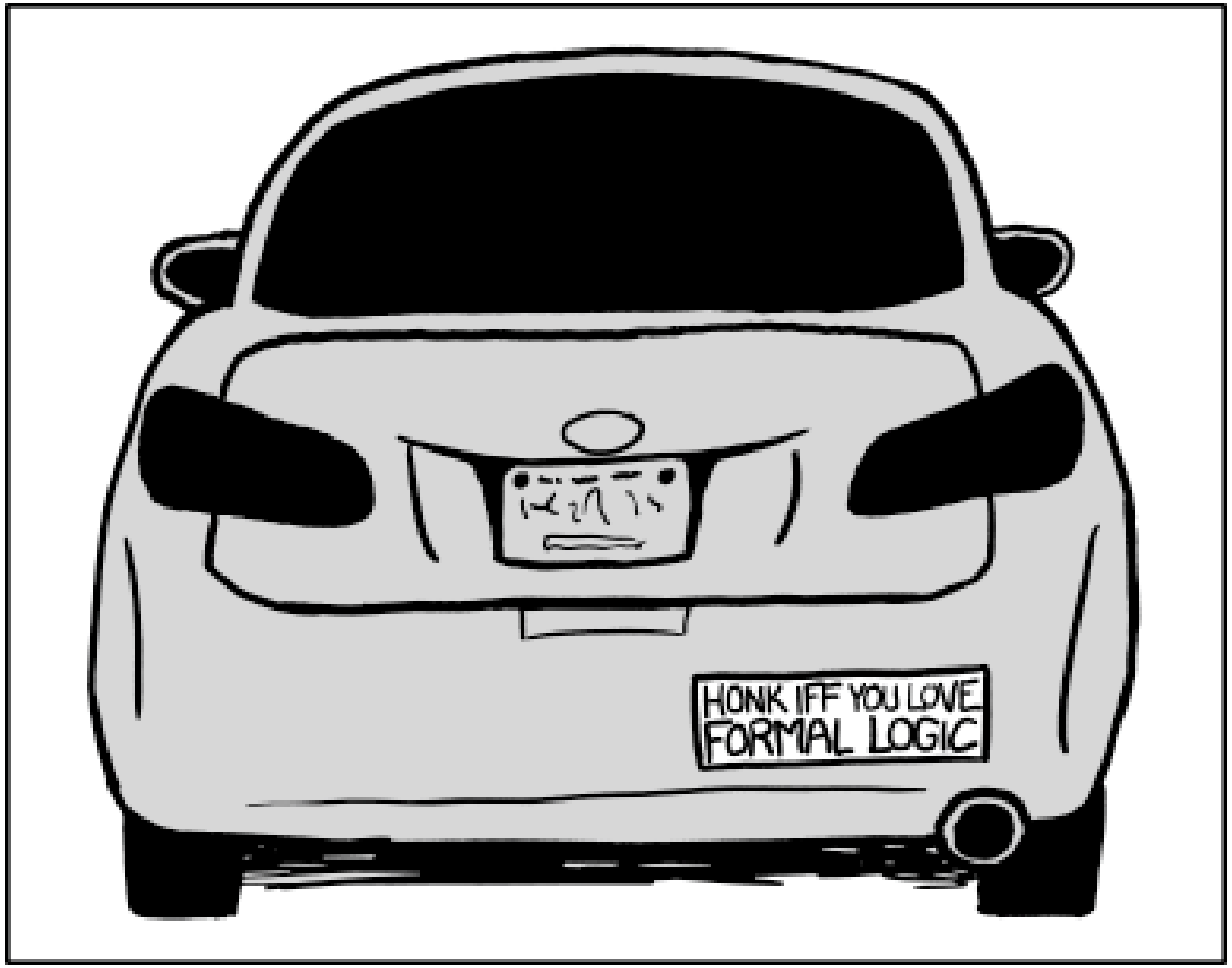
- Order: $x_1 < x_1' < x_2 < x_2' <$ ... ➡ size of BDD is $3 * n + 2$ nodes
- Order: $x_1 < x_2 < \cdots < x_1{}' < x_2' <$ ... ➡ size of BDD is $3 * 2^n - 1$ nodes

Learning Outcomes

After this lecture…

1. students can define and explain BDDs.
    - BDD = reduced and ordered binary decision diagram
    - Define and explain its elements and their meaning
2. students can represent a formula in propositional logic as BDD.
3. students can derive the formula that is represented by a BDD.
4. students can state properties of BDDs.
    - advantages, disadvantages

Thank You



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