Logic and Computability SS24, Assignment 4

Due: 15. 05. 2024, 23:59

SOLUTION

1 Natural Deduction for Predicate Logic

For each of the following sequents, either provide a natural deduction proof, or a counterexample that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/ intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

1. [2 points] $\forall x\ (P(x) \land Q(x)) \quad \vdash \quad \exists x\ (P(x) \lor Q(x))$ 2003.tex

Solution

1.	$\forall x\ (P(x) \land Q(x))$	prem
2.	$P(x_0) \land Q(x_0)$	$\forall$e1
3.	$P(x_0)$	$\land$e2
4.	$P(x_0) \lor Q(x_0)$	$\lor$i3
5.	$\exists x\ (P(x) \lor Q(x))$	$\exists$i4

2003_sol.t

2. [3 points] $\exists x\ \neg P(x) \quad \vdash \quad \neg\forall x\ P(x)$. 2008.tex

Solution

1.	$\exists x\ \neg P(x)$	premise
2.	$\forall x\ P(x)$	assumption
3.	$P(x_0)$	$\forall_e 2$
4.	$\neg P(x_0)$	assumption fresh x_0
5.	$\bot$	$\neg_e 3, 4$
6.	$\bot$	$\exists_e 1, 3-5$
7.	$\neg\forall x\ P(x)$	$\neg_i 2-5$

1013_sol.t

2007.tex

3. [2 points] $\exists x\ (P(x) \vee Q(x)) \quad \vdash \quad \exists x\ P(x) \vee \exists x\ Q(x)$

Solution

1.		$\exists x\ (P(x) \vee Q(x))$	prem.
2.	x_0	$P(x_0) \vee Q(x_0)$	ass.
3.		$P(x_0)$	ass.
4.		$\exists x\ P(x)$	∃i3
5.		$\exists x\ P(x) \vee \exists x\ Q(x)$	∨i4
6.		$Q(x_0)$	ass.
7.		$\exists x\ Q(x)$	∃i6
8.		$\exists x\ P(x) \vee \exists x\ Q(x)$	∨i7
9.		$\exists x\ P(x) \vee \exists x\ Q(x)$	$\vee$e$2, 3-8$
10.		$\exists x\ P(x) \vee \exists x\ Q(x)$	$\exists$e$1, 2-9$

2007_sol.t

0012.tex

4. [2 points] $\exists x\ \neg P(x), \exists x\ \neg Q(x) \quad \vdash \quad \exists x\ (\neg P(x) \wedge \neg Q(x))$

Solution

This sequent is not provable.
Model $\mathcal{M}$:

$$\begin{aligned} \mathcal{A} &= \{a, b\} \\ P^{\mathcal{M}} &= \{a\} \\ Q^{\mathcal{M}} &= \{b\} \end{aligned}$$

$\mathcal{M} \models \exists x \, \neg P(x), \exists x \, \neg Q(x)$
$\mathcal{M} \not\models \exists x \, (\neg P(x) \wedge \neg Q(x))$

0012_sol.t

5. [3 points] $\forall x \, (P(x) \vee Q(x)), \quad \forall x \, (\neg P(x)) \quad \vdash \quad \forall x \, (Q(x))$ 1003.tex

Solution

1.	$\forall x \, (P(x) \vee Q(x))$	premise
2.	$\forall x \, (\neg P(x))$	premise
3.		fresh x_0
4.	$P(x_0) \vee Q(x_0)$	$\forall_e 1 \; x_0$
5.	$\neg P(x_0)$	$\forall_e 2$
6.	$P(x_0)$	assumption
7.	$\perp$	$\neg_e 5, 6$
8.	$Q(x_0)$	$\perp_e 7$
9.	$Q(x_0)$	assumption
10.	$Q(x_0)$	$\vee_e 4, 6-8, 9$
11.	$\forall x \; Q(x)$	$\forall_i 3-10$

1003_sol.t

6. [3 points] $\forall a \forall b \, (P(a) \wedge Q(b)) \quad \vdash \quad \forall a \exists b \, (P(a) \vee Q(b))$ 0005.tex

Solution

1.		$\forall a \forall b\ (P(a) \wedge Q(b))$	prem.
2.	t	$\forall b\ (P(s) \wedge Q(b))$	$\forall$e 1
3.		$P(s) \wedge Q(t)$	$\forall$e 2
4.		$P(s)$	$\wedge e_1$ 3
5.		$P(s) \vee Q(t)$	$\vee i_1$ 4
6.		$\exists b\ (P(s) \vee Q(b))$	$\exists$i 5
7.		$\forall a \exists b\ (P(a) \vee Q(b))$	$\forall$i 2-6

0005_sol.t