Logic and Computability SS24, Assignment 2

Due: 10. 04. 2024, 23:59

SOLUTION

1 Natural Deduction for Propositional Logic

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/ intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

1. [2 points] $x \wedge (y \wedge z) \;\vdash\; (x \wedge y) \wedge z$

Solution

1.	$x \wedge (y \wedge z)$	premise
2.	x	$\wedge_e 1 1$
3.	$y \wedge z$	$\wedge_e 2 1$
4.	y	$\wedge_e 1 3$
5.	z	$\wedge_e 2 3$
6.	$x \wedge y$	$\wedge_i 2, 4$
7.	$(x \wedge y) \wedge z$	$\wedge_i 5, 6$

2. [3 points] $p \wedge q \vee r \;\vdash\; (p \vee r) \wedge (q \vee r)$

Solution

This sequent is provable.

1.	$(p \wedge q) \vee r$	premise	
2.	$p \wedge q$	assumption	
3.	p	$\wedge_e 12$	
4.	$p \vee r$	$\vee_i 13$	
5.	q	$\wedge_e 22$	
6.	$q \vee r$	$\vee_i 15$	
7.	$(p \vee r) \wedge (q \vee r)$	$\wedge_i 4, 6$	
8.	r	assumption	
9.	$p \vee r$	$\vee_i 28$	
10.	$q \vee r$	$\vee_i 28$	
11.	$(p \vee r) \wedge (q \vee r)$	$\wedge_i 9, 10$	
12.	$(p \vee r) \wedge (q \vee r)$	$\vee_e 1, 2-7, 8-11$	

3. [3 points]

(a) $\vdash \neg(\neg p \vee q) \vee p$

(b) $\vdash \neg p \vee (\neg q \vee p)$

Solution

(a) This sequent is not provable.

$\mathcal{M} : p = F, q = T$
$\mathcal{M} \models \mathbf{T}$
$\mathcal{M} \not\models \neg(\neg p \vee q) \vee p$

(b)

1.	$p \vee \neg p$	LEM
2.	$\neg p$	assumption
3.	$\neg p \vee (\neg q \vee p)$	$\vee_i 1\ 2$
4.	p	assumption
5.	$\neg q \vee p$	$\vee_i 3\ 4$
6.	$\neg p \vee (\neg q \vee p)$	$\vee_i 1\ 5$
7.	$\neg p \vee (\neg q \vee p)$	$\wedge_e 1$, 2-3, 4-6

4. [3 points] $(p \to q) \land (q \to r), p \;\vdash\; \neg\neg r \land \neg p$

Solution

This sequent is not provable, counter example:
$\mathcal{M} : p = T, q = T, r = T$
$\mathcal{M} \models (p \to q) \land (q \to r), p$
$\mathcal{M} \not\models \neg\neg r \land \neg p$

5. [4 points] $\neg(q \land p) \;\vdash\; \neg q \lor \neg p$

Solution

This sequent is provable.

1.	$\neg(q \land p)$	premise
2.	$q \lor \neg q$	LEM
3.	$p \lor \neg p$	LEM
4.	q	assumption
5.	p	assumption
6.	$q \land p$	$\land_i 4, 5$
7.	$\bot$	$\neg_e 1, 6$
8.	$\neg q \lor \neg p$	$\bot_e 7$
9.	$\neg p$	assumption
10.	$\neg q \lor \neg p$	$\lor_i 2 9$
11.	$\neg q \lor \neg p$	$\lor_e 3, 5-8, 9-10$
12.	$\neg q$	assumption
13.	$\neg q \lor \neg p$	$\lor_i 1 12$
14.	$\neg q \lor \neg p$	$\lor_e 2, 4-11, 12-13$

Solution

Alternative Solution:

1.	$\neg(q \wedge p)$	premise
2.	$\neg(\neg q \vee \neg p)$	assumption
3.	$\neg q$	assumption
4.	$\neg q \vee \neg p$	$\vee_i 13$
5.	$\bot$	$\neg_e 2, 4$
6.	q	PBC $3-5$
7.	$\neg p$	assumption
8.	$\neg q \vee \neg p$	$\vee_i 27$
9.	$\bot$	$\neg_e 2, 8$
10.	p	PBC $7-9$
11.	$q \wedge p$	$\wedge_i 6, 10$
12.	$\bot$	$\neg_e 1, 11$
13.	$\neg q \vee \neg p$	PBC $2-12$

Alternative solution:

1.	$\neg(q \wedge p)$	premise
2.	$\neg q \vee q$	LEM
3.	$\neg q$	assumption
4.	$\neg q \vee \neg p$	$\vee_i 13$
5.	q	assumption
6.	p	assumption
7.	$q \wedge p$	$\wedge_i 5, 6$
8.	$\bot$	$\neg_e 1, 7$
9.	$\neg p$	$\neg_i 6-8$
10.	$\neg q \vee \neg p$	$\vee_i 29$
11.	$\neg q \vee \neg p$	$\vee_e 2, 3-4, 5-10$