Logic and Computability SS24, Assignment 1

Due: 20. 03. 2024, 23:59

SOLUTION

The DPLL-Algorithm

For the following exercises use the DPLL algorithm (including Boolean Constraint Propagation (BCP), pure literals, and conflict-driven clause learning) to check on paper, if the following CNF formulas are satisfiable.
If the formula is satisfiable, give a satisfying model, else show a complete resolution proof for the formula's unsatisfiability.

- Write down all the steps of the DPLL algorithm,
- draw the conflict graphs,
- and state the resolution proofs for all learned clauses.

Rules:

- When resolving a conflict, only undo the last decision.
- Choose variables for decisions, BCP and pure literals in alphabetical order, starting with the *negative* phase ($\neg a > a > \neg b > b ...$).
- Always try to perform BCP first, before checking for pure literals, before making a decision.

1. [3 points] Use the DPLL algorithm with the rules as described above to check whether the following formula in CNF is satisfiable

   Clause 1: $\{a, \neg b, c\}$

   Clause 2: $\{b, \neg c, d\}$

   Clause 3: $\{a, \neg b\}$

   Clause 4: $\{a, c\}$

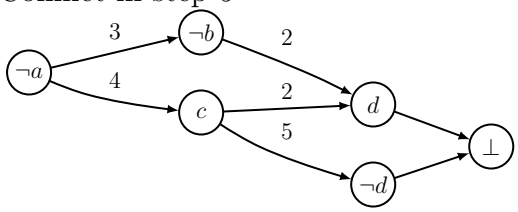
Clause 5: $\{\neg c, \neg d\}$

Clause 6: $\{\neg a, c\}$

Solution

Step	1	2	3	4	5
Decision Level	0	1	1	1	1
Assignment	-	$\neg a$	$\neg a, \neg b$	$\neg a, \neg b, c$	$\neg a, \neg b, c, \neg d$
Cl. 1: $a, \neg b, c$	$a, \neg b, c$	$\neg b, c$	✓	✓	✓
Cl. 2: $b, \neg c, d$	$b, \neg c, d$	$b, \neg c, d$	$\neg c, d$	d	$\{\}$ ✗
Cl. 3: $a, \neg b$	$a, \neg b$	$\neg b$	✓	✓	✓
Cl. 4: a, c	a, c	c	c	✓	✓
Cl. 5: $\neg c, \neg d$	$\neg c, \neg d$	$\neg c, \neg d$	$\neg c, \neg d$	$\neg d$	✓
Cl. 6: $\neg a, c$	$\neg a, c$	✓	✓	✓	✓
BCP	-	$\neg b$	c	$\neg d$	-
PL	-	-	-	-	-
Decision	$\neg a$	-	-	-	-

Conflict in step 5



$$\dfrac{\dfrac{\dfrac{\dfrac{2.\ b \vee \neg c \vee d \qquad 5.\ \neg c \vee \neg d}{b \vee \neg c} \qquad 3.\ a \vee \neg b}{\neg c \vee a} \qquad 4.\ a \vee c}{a}$$

Step	6	7	8	9	10
Decision Level	0	0	0	0	0
Assignment	-	a	a, c	$a, c, \neg d$	$a, c, \neg d, b$
Cl. 1: $a, \neg b, c$	$a, \neg b, c$	✓	✓	✓	✓
Cl. 2: $b, \neg c, d$	$b, \neg c, d$	$b, \neg c, d$	b, d	b	✓
Cl. 3: $a, \neg b$	$a, \neg b$	✓	✓	✓	✓
Cl. 4: a, c	a, c	✓	✓	✓	✓
Cl. 5: $\neg c, \neg d$	$\neg c, \neg d$	$\neg c, \neg d$	$\neg d$	✓	✓
Cl. 6: $\neg a, c$	$\neg a, c$	c	✓	✓	✓
Cl. 7: a	a	✓	✓	✓	✓
BCP	a	c	$\neg d$	b	-
PL	-	-	-	-	-
Decision	-	-	-	-	SAT

2. [2 points] Use the DPLL algorithm with the rules as described above to check whether the following formula in CNF is satisfiable

Clause 1: $\{a, b, c\}$

Clause 2: $\{\neg b, \neg c, e\}$

Clause 3: $\{b, e\}$

Clause 4: $\{b, \neg d\}$

Clause 5: $\{\neg c, d\}$

Clause 6: $\{\neg c, e\}$

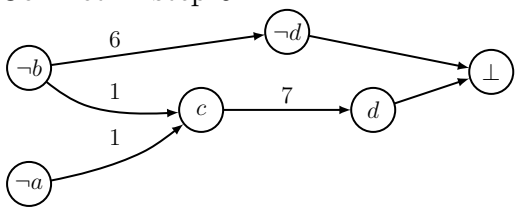
Clause 7: $\{\neg a, \neg b, \neg c\}$

Clause 8: $\{a, c, \neg e\}$

Solution

Step	1	2	3	4	5
Decision Level	0	1	2	2	2
Assignment	-	$\neg a$	$\neg a, \neg b$	$\neg a, \neg b, c$	$\neg a, \neg b, c, \neg d$
Cl. 1: a, b, c	a, b, c	b, c	c	✓	✓
Cl. 2: $\neg a, \neg b, \neg c$	$\neg a, \neg b, \neg c$	✓	✓	✓	✓
Cl. 3: $a, c, \neg e$	$a, c, \neg e$	$c, \neg e$	$c, \neg e$	✓	✓
Cl. 4: $\neg b, \neg c, e$	$\neg b, \neg c, e$	$\neg b, \neg c, e$	✓	✓	✓
Cl. 5: b, e	b, e	b, e	e	e	e
Cl. 6: $b, \neg d$	$b, \neg d$	$b, \neg d$	$\neg d$	$\neg d$	✓
Cl. 7: $\neg c, d$	$\neg c, d$	$\neg c, d$	$\neg c, d$	d	$\{\}$ ✗
Cl. 8: $\neg c, e$	$\neg c, e$	$\neg c, e$	$\neg c, e$	e	e
BCP	-	-	c	$\neg d$	-
PL	-	-	-	-	-
Decision	$\neg a$	$\neg b$	-	-	-

Conflict in step 5



$$\dfrac{\dfrac{7.\ \neg c \vee d \qquad 1.\ a \vee b \vee c}{d \vee a \vee b} \qquad 6.\ b \vee \neg d}{a \vee b}$$

Step	6	7	8	9	10
Decision Level	1	1	1	2	2
Assignment	$\neg a$	$\neg a, b$	$\neg a, b, d$	$\neg a, b, d, \neg c$	$\neg a, b, d, \neg c, \neg e$
Cl. 1: a, b, c	b, c	✓	✓	✓	✓
Cl. 2: $\neg a, \neg b, \neg c$	✓	✓	✓	✓	✓
Cl. 3: $a, c, \neg e$	$c, \neg e$	$c, \neg e$	$c, \neg e$	$\neg e$	✓
Cl. 4: $\neg b, \neg c, e$	$\neg b, \neg c, e$	$\neg c, e$	$\neg c, e$	✓	✓
Cl. 5: b, e	b, e	✓	✓	✓	✓
Cl. 6: $b, \neg d$	$b, \neg d$	✓	✓	✓	✓
Cl. 7: $\neg c, d$	$\neg c, d$	$\neg c, d$	✓	✓	✓
Cl. 8: $\neg c, e$	$\neg c, e$	$\neg c, e$	$\neg c, e$	✓	✓
Cl. 9: a, b	b	✓	✓	✓	✓
BCP	b	-	-	$\neg e$	-
PL	-	d	-	-	-
Decision	-	-	$\neg c$	-	SAT

3. [3 points] Use the DPLL algorithm with the rules as described above to check whether the following formula in CNF is satisfiable

Clause 1: $\{a, b\}$

Clause 2: $\{\neg b, c\}$

Clause 3: $\{\neg a, \neg c\}$

Clause 4: $\{b, c\}$

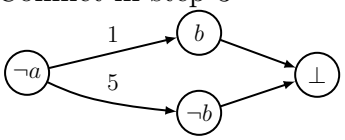
Clause 5: $\{a, \neg b\}$

Clause 6: $\{\neg b, \neg c\}$

Solution

Step	1	2	3
Decision Level	0	1	1
Assignment	-	$\neg a$	$\neg a, \neg b$
Cl. 1: a, b	a, b	b	{} ✗
Cl. 2: $\neg b, c$	$\neg b, c$	$\neg b, c$	✓
Cl. 3: $\neg a, \neg c$	$\neg a, \neg c$	✓	✓
Cl. 4: b, c	b, c	b, c	c
Cl. 5: $a, \neg b$	$a, \neg b$	$\neg b$	✓
Cl. 6: $\neg b, \neg c$	$\neg b, \neg c$	$\neg b, \neg c$	✓
BCP	-	$\neg b$	-
PL	-	-	-
Decision	$\neg a$	-	-

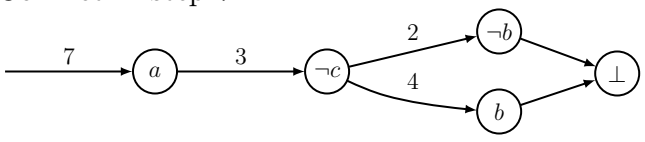
Conflict in step 3



$$\frac{1.\ a \vee b \qquad 5.\ a \vee \neg b}{a}$$

Step	4	5	6	7
Decision Level	0	0	0	0
Assignment	-	a	$a, \neg c$	$a, \neg c, \neg b$
Cl. 1: a, b	a, b	✓	✓	✓
Cl. 2: $\neg b, c$	$\neg b, c$	$\neg b, c$	$\neg b$	✓
Cl. 3: $\neg a, \neg c$	$\neg a, \neg c$	$\neg c$	✓	✓
Cl. 4: b, c	b, c	b, c	b	{} ✗
Cl. 5: $a, \neg b$	$a, \neg b$	✓	✓	✓
Cl. 6: $\neg b, \neg c$	$\neg b, \neg c$	$\neg b, \neg c$	✓	✓
Cl. 7: a	a	✓	✓	✓
BCP	a	$\neg c$	$\neg b$	-
PL	-	-	-	-
Decision	-	-	-	UNSAT

Conflict in step 7



$$\frac{\dfrac{\dfrac{2.\ \neg b \vee c \qquad 4.\ b \vee c}{c} \qquad 3.\ \neg a \vee \neg c}{\neg a} \qquad 7.\ a}{\bot}$$

4. [3 points] Construct a reduced ordered binary decision diagram (ROBDD) for the formula

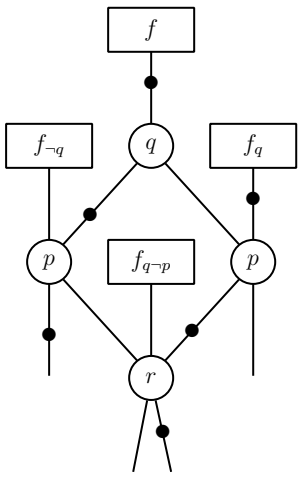
$$f = (p \vee q) \wedge \neg(p \wedge q) \wedge r$$

using *variable order* $q < p < r$. Use complemented edges and a node for `true` as the only constant node. To simplify drawing, you may assume that *dangling edges* point to the constant node. Write down all cofactors that you compute to obtain the final result and mark them in the graph.

Solution

$f = (((p \vee q) \wedge \neg(p \wedge q)) \wedge r)$
$f_q = (\neg(p \wedge q) \wedge r)$
$f_{qp} = \bot$
$f_{q\neg p} = r$
$f_{q\neg pr} = \top$
$f_{q\neg p\neg r} = \bot$
$f_{\neg q} = (p \wedge r)$
$f_{\neg qp} = f_{q\neg p}$
$f_{\neg q\neg p} = \bot$

The final ROBDD:



5. [3 points] Construct a reduced ordered binary decision diagram (ROBDD) for the formula

$$(a \vee \neg b) \wedge \neg(c \vee d) \vee (a \wedge b),$$

using *variable order* $a < b < c < d$. Use complemented edges and a node for `true` as the only constant node. To simplify drawing, you may assume that *dangling edges* point to the constant node. Write down all cofactors that you compute to obtain the final result and mark them in the graph.

Solution

$f = (((a \vee \neg b) \wedge \neg(c \vee d)) \vee (a \wedge b))$

$f_a = (\neg(c \vee d) \vee b)$

$f_{ab} = \top$

$f_{a\neg b} = \neg(c \vee d)$

$f_{a\neg bc} = \bot$

$f_{a\neg b\neg c} = \neg d$

$f_{a\neg b\neg cd} = \bot$

$f_{a\neg b\neg c\neg d} = \top$

$f_{\neg a} = (\neg b \wedge \neg(c \vee d))$

$f_{\neg ab} = \bot$

$f_{\neg a\neg b} = f_{a\neg b}$

The final ROBDD: