Electromagnetic fields and Mechanical Motion

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Abstract

In particle physics gauge theory provides a powerful insight into the electromagnetic interaction and its relationship with the weak interaction. Here we translate some of the gauge theory arguments for electromagnetism into the notation of second quantization, where it will be of immediate use to the solid state physicist. We try to bring out the relationship between electromagnetic fields and mechanical motion. For example, the magnetic field $\underline{B}$ is related to a rotation in our formalism. We later speculate on the role of Mach's principle with respect to the magnetic field.

1 Introduction

To a physicist there are three sorts of knowledge: what he knows; what he does not know but believes that he could measure at least in principle; and what he believes that he can never know. Most theoretical physics is based upon the first category, what is known. Most experimental physics is addressed towards the second category, what one could measure. But one would perhaps expect to make little headway with work based upon the third category, what can never be known. Yet it is precisely this sort of work which can be of the most fundamental importance and which can be most illuminating about the nature of our world.

For example, the theory of relativity is based upon the assumption that one can never know one's absolute velocity in space - a remarkable edifice built upon the seemingly insecure foundation of our lack of knowledge about our world. General relativity is likewise built upon the assumption that one can never distinguish between a uniform gravitational field and an acceleration. Of particular interest in this chapter is the gauge theory of the electromagnetic interaction. This assumes as a basis that the absolute phase of any quantum-mechanical parameter is unobservable, and from this basis it infers the existence of the electromagnetic field and the Maxwell equations governing it. This theory throws new light upon the electromagnetic interaction by emphasizing the importance of certain relationships, and aesthetically it is a great deal more pleasing than rote learning of the Maxwell equations. For this reason time is devoted to it in this chapter, though it must be realized that any observable results which may follow from this approach could have been derived by application of the usual electromagnetic equations. Gauge theory does of course have a powerful predictive power when it is extended to cover the weak interaction, and it is this route which has led to the unification of the weak and electromagnetic interactions, although this extension is not discussed here.

In this chapter we shall discuss part of the gauge theory of the electromagnetic interaction. We shall translate some of the notation into the formalism of second quantization. This formalism will be of use in the later chapters. For example, given a quantum-mechanical description of a system, the formalism will enable us to write down the description of an otherwise identical system which has been subjected to an electromagnetic field. Certain approximations are involved here.

In this chapter we shall also compare and contrast the effects of the electromagnetic field, with the effects of mechanical motion. We shall see that there are similarities between the Lorentz transformation and the electromagnetic gauge transformation. Neither is ever observable in its own right. However, a rotation velocity is intrinsically observable, and in our formalism we analyse rotation in terms of a set of local Lorentz transformations where the velocity $\underline{u}$ varies in space. The quantity curl $\underline{u}$ is observable. Likewise, in electromagnetism the magnetic vector potential $\underline{A}$ cannot be intrinsically observed, whilst the magnetic field (curl $\underline{A}$) can be observed without reference to systems outside. We try to bring out the connection between these in this chapter.

As was originally suggested by Mach, it [1] is not pleasing aesthetically to suppose that there is some absolute frame of non-rotation. It would instead be preferable to have a formalism in which all rotation is relative to some large massive bodies, and the fixed stars have been suggested as a source for some 'rotation field' although formalisms based upon this have not had a great deal of success. In view of the similarities between rotation fields and magnetic fields, we discuss whether Mach's question should be extended to magnetism: 'is there some absolute frame of zero magnetic field ? '

This last discussion related to Mach's principle is included because it was the posing of questions upon these lines which was the driving force behind the development of most of the theory in this dissertation. The questions themselves remain obstinately unanswered, although the theory which was developed in order to answer them has provided the catalyst for the development of much of the work contained in later chapters of this dissertation. The questions themselves remain as valid and interesting as ever.

Before continuing with the details of the analysis outlined above, the reader is directed towards Appendix 1 of this chapter. There we quote an extract from a series of lectures on gauge theory, given from the point of view of a particle physicist. Much of the work in this chapter was inspired by the economical and powerful arguments there set forth, and the next few sections are an attempt to translate some of these arguments into the notation of second quantization.

2 Unobservable transformation in second quantization

In the formalism of second quantization, (3) there are used two basic quantities: the state-vector $|\psi\rangle$ which describes the state of a quantum-mechanical system in terms of the occupation of eigenvectors; and the operators $\hat{O}$ which act upon the state-vectors. A useful combination of these can be made for Hermitean operators $\hat{O}_H$ (that is, $\hat{O}_H^+ = \hat{O}_H$) to form quantities called 'observables', of form $\langle O\rangle = \langle\psi|\hat{O}_H|\psi\rangle$.

If a transformation is to be unobservable, then the action of the transformation may change either or both of the state-vectors $|\psi\rangle$ and the operators $\hat{O}$, but the transformation must leave unchanged the observables $\langle O\rangle$. This condition puts several constraints upon the type of transformation allowed, and we now consider these constraints.

Let the transformation which is to be unobservable be called T, so that its effect upon a state vector is:

$$|\psi\rangle \rightarrow T|\psi\rangle \qquad (2.1)$$

The first constraint upon the form of T may be seen by noting that the unity operator I is Hermitean, and so the quantity $\langle\psi|I|\psi\rangle$ must be conserved under the transformation. If our notation is to be consistent and convenient, the operator I must be unchanged under the transformation, and so we deduce that:

$$\langle\psi|I|\psi\rangle \rightarrow \langle\psi|T^+ \; I \; T|\psi\rangle = \langle\psi|I|\psi\rangle \qquad (2.2)$$

and therefore that the transformation operator T must obey the condition that $T^+T = 1$, i.e. T is a unitary operator.

The second constraint upon the transformation may be seen by considering the effect of the transformation upon an observable $\langle O \rangle$. Writing the transformation of observables:

$$\langle O \rangle = \langle \Psi | \hat{O}_H | \Psi \rangle \rightarrow \langle \Psi | T^+ \; \hat{O}'_H \; T | \Psi \rangle$$
$$= \langle \Psi | \; \hat{O}_H \; | \Psi \rangle \qquad (2.3)$$

then it follows that the transformation of Hermitean operators must be:

$$\hat{O}'_H = T \; \hat{O}_H \; T^+ \qquad (2.4)$$

3 The Spatial Translation

We now illustrate these properties by considering the spatial translation operator which is designed to cause a complete system to be translated in space through a constant vector $\underline{r}$. It is assumed that it is impossible to determine one's absolute position in space, and so this translation operator should be unobservable.

We define the translation operator in terms of its effect upon the simple state-vector:

$$|\psi\rangle \quad = \quad \hat{\psi}^+(\underline{x}_1)\ \hat{\psi}^+(\underline{x}_2)\ \ldots\ \hat{\psi}^+(\underline{x}_N)\ |0\rangle \tag{3.1}$$

where the $\hat{\psi}^+$ operators create a particle at a given position in space, and $|0\rangle$ is the vacuum state-vector. This state-vector contains N particles at positions x_1 through to x_N, and it will be clear that the general N particle state-vector can be constructed by integrating up over $x_1 .. x_N$ states of the form (3.1). The effect of the translation operator $T(\underline{r})$ upon this state-vector must be to move all the particles through the vector $\underline{r}$, i.e.

$$T(\underline{r})\,|\psi\rangle = \hat{\psi}^+(\underline{x}_1+\underline{r})\ \ldots\ \hat{\psi}^+(\underline{x}_N+\underline{r})\ |0\rangle \tag{3.2}$$

In order to obtain an expression for the operator $T(\underline{r})$, we define the total momentum operator:

$$\hat{P} \quad = \quad \sum_k \hat{N}_k\, k \tag{3.3}$$

where $\hat{N}_k$ is the counting operator for the number occupation of the exact Fourier state k. Writing the field operator in the form:

$$\hat{\psi}^+(x_i) = \sum_k a_k^+\ e^{-ik\cdot x_i} \tag{3.4}$$

we can observe that the following identity holds:

$$e^{-i\hat{P}.\underline{r}}\ \hat{\psi}^{+}(x_i) \quad = \quad \sum_{k} a_k^{+}\ e^{-ik.(\underline{x}_i+\underline{r})}\ e^{-i\hat{P}.\underline{r}} \tag{3.5}$$

$$= \quad \hat{\psi}^{+}(\underline{x}_i+\underline{r})\ e^{-i\hat{P}.\underline{r}} \tag{3.6}$$

The operator $e^{-i\hat{P}.\underline{r}}$ can clearly be identified with the T($\underline{r}$) provided that we assume that the effect of this operator upon the vacuum state is null. It is conventional to assume that the total momentum of the vacuum state is zero, in which case our operator clearly leaves it unchanged as required; we shall also adopt this convention. Hence we may write that

$$T(\underline{r}) \quad = \quad e^{-i\hat{P}.\underline{r}} \tag{3.7}$$

An interpretation of the meaning of the translation operator may be made by considering a system of particles, and a detector which measures some property of the system of particles. In quantum mechanics, the system of particles is described by the state-vector $|\psi\rangle$, and the detector by an operator $\hat{O}$. Application of the translation (2.1) to the state-vector causes the particles all to be moved through the vector $\underline{r}$; application of (2.4) to the operators causes the detector to be moved through the same vector $\underline{r}$; so that both together change none of the distances of any of the particles relative to the detector and hence the complete translation is not observable.

Thus far, we have only considered translations which are never observable; and if this were all that there were to the study then the subject would be rather empty to say the least. We have set up a formalism with so much symmetry between particles and detectors that the formalism has no predictive power for actual experiments. If we were to succeed in breaking this symmetry, however, then something useful might result, and we try to do this now for the translation operator.

What would be the effect of applying the translation (2.4) to the detector in our system, whilst leaving the system itself unchanged ? In effect this corresponds to applying half of the unobservable translation, and so we might expect there to be some measurable result. If originally the detector measured some quantity which were a function of position, say the probability of finding a particle at position $\underline{x}_0$, then the translated operator would measure the same quantity at a position which is vector $\underline{r}$ away from the original measurement position, in our example the probability of finding a particle at position $\underline{x}_0 + \underline{r}$. To put this statement into more mathematical terms, we may say:

$$\langle O(\underline{r}) \rangle = \langle \psi | \; T(\underline{r}) \; \hat{O}(\underline{0}) \; T^{+}(\underline{r}) \; | \psi \rangle \qquad (3.8)$$

where $\langle O(\underline{r}) \rangle$ is the expectation of some observable at position r , $\hat{O}(\underline{0})$ is the operator for that observable at position $\underline{0}$.

This example has been worked through in order to exhibit the process of generalizing from an unobservable transformation to something more useful; further examples will be given later in this chapter.

Two points should be noted here about the equation (3.8). Firstly, if a state-vector has zero total momentum, i.e. $\hat{p}|\psi\rangle = 0$, then from (3.7) it is translationally invariant in the sense that $T^{+}|\psi\rangle = |\psi\rangle$, and so from (3.8) all expectations $\langle O(r) \rangle$ of the above form are independent of position. This remarkable general result will be of use when we come to consider superfluids in the later chapters. Secondly, the equation (3.7) can be regarded as an extension of the familiar relationship in which the Hamiltonian is the generator of infinitesimal time translations, which we will write in our notation as:

$$T(t) = e^{i\hat{H}t} \qquad (3.9)$$

where $\hat{H}$ is the Hamiltonian, which is regarded as the time component of the operator for the 4-momentum of the system. This point will be considered in more detail later in this chapter.

4 Galilean Transformations

In this section we try to develop a formalism to describe the effects of a Galilean transformation upon a quantum-mechanical system. In other words, given a description of a system at rest, we wish to be able to describe an otherwise identical system, which has been set in motion with uniform velocity $\underline{u}$. We limit ourselves to the non-relativistic case where $u \ll c$.

We shall see that the formalism described in this chapter is similar in form to that describing the spatial translation. In particular, one can never determine one's absolute velocity, so that the full Galilean transformation (i.e. where both system and observer are set moving with velocity $\underline{u}$) is unobservable.

Suppose that a system at rest is described by the state-vector

$$|\psi\rangle = a^{+}_{k1}\, a^{+}_{k2} \cdots a^{+}_{kN}\, |0\rangle \qquad (4.1)$$

We now wish to describe an otherwise identical system, which has been set in motion with uniform velocity $\underline{u}$. Our description will be from the point of view of an observer at rest. The momenta of the states $k_1 \cdots k_N$ will be changed by the velocity boost. The amount of the change in each of the k-vectors k_i depends upon the mass m_i of a particle in that state. In a simple fermionic system, each of the masses m_i would equal the rest mass m_0 of a particle, but in practise there will be corrections due to kinetic and interaction mass-energies. The momentum of a state i will be increased by an amount $m_i\,\underline{u}$, so that the Galilean transformation $L(u)$ has the net effect:

$$L(u)\,|\psi\rangle = a^{+}_{k1+m1\cdot u} \cdots a^{+}_{kN+mN\cdot u}\, |0\rangle \qquad (4.2)$$

where we have chosen units $\hbar = 1$.

To proceed further, let us define the operator for total mass $\hat{M}$:

$$\hat{M} = \sum_i m_i \hat{N}_i \tag{4.3}$$

where $\hat{N}_i$ is the counting operator for particles in state i. Clearly the transformed state (4.2) possesses more expectation momentum than the state at rest, (4.1), by an amount $\langle \hat{M} u \rangle$. It follows from this observation (or alternatively from direct substitution) that the commutator of the momentum operator $\hat{P}$ with the transformation L(u) is:

$$[\hat{P}, L(u)] = \hat{M} u L(u) \tag{4.4}$$

We have remarked earlier that if the observer is set in motion at the same velocity as the system, then the whole transformation must be unobservable - all observables must be unchanged. It therefore follows from (2.4) that the effect of setting the observer in motion with uniform velocity $\underline{u}$ must be described by changing the operators thus:

$$\hat{O}_H \longrightarrow L(u)\, \hat{O}_H\, L^{+}(u) \tag{4.5}$$

Any measurements the moving observer performs on the system must yield the same results as those of a stationary observer on a stationary system.

It is possible to re-write (4.5) in a more illuminating way. Since the transformation L(u) does not affect the position, number or any other scalar properties about the system, then L(u) commutes with these quantities and (4.5) leaves their operators unchanged. This is not true for the momentum operator $\hat{P}$. Using (4.4) it follows that (4.5) may be re-written:

$$\hat{P} \longrightarrow \hat{P} - \hat{M} u \tag{4.6}$$

In fact, (4.6) may be written more fully as the set of equations $\hat{p}_i \rightarrow \hat{p}_i - \hat{m}_i u$, but we shall not be concerned here with this microscopic form of the equation.

We now consider: in what ways can a Galilean transformation be modified so that it does yield observable results ? There are two ways.

The first has already been discussed. Suppose that the transformation is applied to the system but not to the observer. In other words, the system is given a velocity boost whereas the observer remains stationary. The momenta measured by the observer are all changed, so that the unobservable transformation has been generalized to an observable "field".

The second method is of more interest, and appears not to involve observers at all. Suppose that the system is set into rotation. Locally, at each point in space there has been a galilean transformation, and it is a good first approximation simply to use the transformation to obtain the local state-vector. This would be expected to be a good approximation if all significant interactions were short-range. But it turns out that even in this case, there are serious problems. Locally, the momentum is changed by the velocity boost. The angular momentum $\oint \underline{p} \cdot d\underline{l} \;/\; (2\pi)$ is therefore changed. If the system was in a state of quantized angular momentum then when it has been set into rotation it cannot remain in this same state.

The observable effects of rotation of quantum-mechanical systems will be discussed in detail in later chapters.

5 The Galilean Transformation as a low-velocity Lorentz transformation

We may analyze the Galilean transformation in a different way, performing a Lorentz transformation and taking the limit of low velocity. This route is illuminating because it provides an algorithm for calculating the mass $\hat{M}$ which appears in the transformation in equation (4.3)

Consider a translation in both space and time through some constant vector (t,r). The form of this translation can be obtained from (3.7) and (3.9) and it may be written :

$$T(t,r) = e^{i(\hat{H}t - \hat{\underline{P}}.\underline{r})} \tag{5.1}$$

$$= e^{i\hat{P}^{\mu}x_{\mu}} \tag{5.2}$$

where in (5.2) we have written the four-vector x_{μ} in place of the translation vector $(t,\underline{r})$, and we have introduced the notation $\hat{P}^{\mu} = (\hat{H}, -\hat{\underline{P}})$. Since the x_{μ} transform under a Lorentz transformation like a 4-vector, and T(t,r) cannot change under a Lorentz transformation, then the quantity $\hat{P}^{\mu}$ must transform like a 4-vector. Knowing the transformation law for $\hat{P}^{\mu}$, we can deduce the way in which the $\hat{\underline{P}}$ transform under a Lorentz transformation to velocity u:

$$\hat{P} \longrightarrow \gamma(\hat{P} - \hat{H}u/c^2) \approx \hat{P} - \hat{H}u/c^2 \tag{5.3}$$

where $\gamma = (1-u^2/c^2)^{-\frac{1}{2}}$ and for low velocity we take the approximation $\gamma = 1$, giving the second expression on the right hand side of (5.3). This low velocity limit represents a Galilean transformation.

Comparison of (5.3) with (4.6) yields the result:

$$\hat{M} = \hat{H}/c^2 \tag{5.4}$$

Of course, the full relativistic energy of the system must be included in the expression for $\hat{H}$, and in particular the rest mass energy must be included in $\hat{H}$.

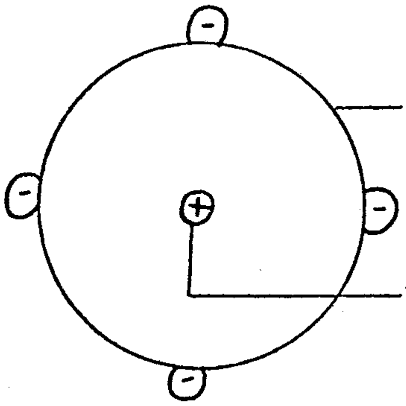
The result (5.4) has one crucial consequence. Suppose that a single particle with mass m_0 and charge e is placed in a region of space with electrostatic potential V. This particle has Hamiltonian energy:

$$\hat{H} = m_0 c^2 + eV \tag{5.5}$$

and so, using (5.4), it appears that the mass m of the particle is modified by the presence of the potential field. In general, if a particle is placed in a potential V, then its mass is increased by $e V / c^2$. Although this corresponds with our ideas of the relativistic equivalence of mass and energy, it does appear to be counter-intuitive: in particle accelerators for example where particles are subjected to very high potentials, no correction is usually made for this effect.

To see the significance of this result in terms of more familiar electric and magnetic fields, we break off for one chapter from our formal exposition of symmetries and the fields derived therefore; we try to discover in what circumstances a particle's mass is affected by a potential field.

6 The mass of a charged particle in a potential field



The diagram shows a particle which has charge $+e$ and mass m_o , which is in the centre of a sphere of negative charge. Ignoring for the moment the potential due to the presence of the particle, the potential inside the sphere is constant, and takes value V.

Imagine in this situation that a small push-rod is attached to the particle, so that the particle may be accelerated and the force required to accelerate the particle may be measured at the other end of the rod. The force required is of course the rate of change of canonical momentum, i.e.

$$\underline{F} = \frac{d}{dt} (m_0 \underline{u} + e \underline{A}) \tag{6.1}$$

If the particle alone is accelerated, then there is no $\underline{A}$ field seen by the particle, and from the symmetry of the sphere there is no electric field; therefore the force is just $\underline{F} = m_o\underline{a}$, where $\underline{a}$ is the acceleration. This result is the same as if there were no sphere of negative charge; i.e. if the particle alone is accelerated then an electrostatic potential has no effect upon the forces required.

What would be the force required, however, if the sphere were also attached to push-rods and caused to accelerate at the same rate as the particle ? The force required for the particle would no longer be the same, because the movement of the charge on the sphere would constitute a current, which would set up a magnetic vector potential field; at the position of the particle there would therefore be a rate of change of magnetic vector potential, or an E field, which would exert a force upon the particle and require a different force to be exerted by the push-rods. The magnitude of the extra forces required are the subject of study of this section.

We first try to evaluate the cannonical momentum associated with the particle when the whole system is moving at velocity $\underline{u}$, so that the force may be calculated using (6.1). The movement of the charges on the sphere at velocity $\underline{u}$ constitutes a current density $\underline{j} = \rho\, \underline{u}$, where ρ is the charge density on the sphere. This produces an $\underline{A}$ field, which can be calculated using the usual Maxwell equations:

$$\nabla^2 \underline{A} = \mu_0 \underline{J} = \mu_0 \rho\, \underline{u}$$

$$= \mu_0 \epsilon_0 \underline{u}\, \nabla^2 V = (\underline{u}/c^2)\, \nabla^2 V \qquad (6.2)$$

In this equation we have used the usual electromagnetism symbols, and c is the velocity of light in a vacuum. Using the boundary condition that at large distance $\underline{A} = \underline{0}$ and $V = 0$, this equation can be integrated up to yield:

$$\underline{A} = \underline{u}\ \ V/c^2 \qquad (6.3)$$

Using (6.3), then we can calculate the canonical momentum of the particle when the whole system is moving at velocity $\underline{u}$. This is:

$$p = m_0 \underline{u} + e\underline{A} = (m_0 + eV/c^2)\, \underline{u} \qquad (6.4)$$

We use equation (4.6) to evaluate the mass m. If a complete system is caused to move at velocity $\underline{u}$ then the canonical momentum is increased by an amount $m\,\underline{u}$. This mass m is just the quantity $(m_o + e\,V / c^2)$ which appears in equation (6.4). The result that $m = m_o + e\,V / c^2$ was derived in section 5 by a different method.

In first quantization, this quantity $\underline{p} = m\,\underline{u}$ is related to the wavelength of the particle. If for example the whole system of particle and negatively charged sphere were put into a Bohr-like orbital around a central point, the quantization condition must be that:

$$\oint \underline{p} \cdot \underline{dl} \quad = \quad m \oint \underline{u} \cdot \underline{dl} \quad = \quad 2n\pi \tag{6.5}$$

where the integral is taken around the central point. The importance of this remark is that it would be wrong to insert the rest mass m_o for the mass relevant here; the corrected mass must be used. In solid state physics one often encounters situations in which a particle is surrounded by a spherical charge density, for example the screening hole surrounding an electron in a metal. In these situations, the mass m above is the parameter relevant to the quantization of the particle.

Before continuing the analysis of the forces required to accelerate our particle, a word of caution. In our definition of the corrected mass m, so that $\underline{p} = m\,\underline{u}$, we have included part of the electromagnetic field in the momentum $\underline{p}$; and it would be incorrect to count again the contribution to the canonical momentum which is due to the $\underline{A}$ field of the moving charges. If there are charges which move at a velocity other than the velocity $\underline{u}$ of the particle, then the canonical momentum of the particle is given by the equation:

$$\underline{p} \quad = \quad m_o\,\underline{u} \; + \; e\,\underline{A} \quad = \quad m\,\underline{u} \; + \; e\,\underline{A}' \tag{6.6}$$

In this equation the quantity $\underline{A}'$ is the extra $\underline{A}$ field which the particle experiences, over and above the field due to the movement of charges at velocity $\underline{u}$. In fact, $\underline{A}'$ is the field which the particle would see if the whole system were subjected to a Galilean transformation to a frame where the particle is stationary. If we have knowledge about the system when it is stationary, and we wish to obtain a description of the moving system, then the second expression of the right hand side of (6.6) is the most convenient form; if however we do not have this knowledge then the first is probably the safest expression to use.

At first glance it might appear that the force required in order to accelerate our particle when the sphere of charge is caused to accelerate at the same rate, is just the rate of change of momentum of the particle. That is, we would expect the force to be:

$$\underline{F} = m\,\underline{a} \qquad \text{(wrong)} \tag{6.7}$$

To see why this cannot be correct, consider the simpler example of a positron and an electron, each in the potential field of the other at potential $\pm V$. The gravitational mass of this combined system is just $2m_o - eV/c^2$, calculated from the rest masses of the particles and the energy absorbed in bringing them together. The force to accelerate this whole system is therefore $(2m_o - eV/c^2)\,\underline{a}$. But from (6.7), the rate of change of momentum associated with the particles is $2m\underline{a} = (2m_o - 2eV/c^2)\,\underline{a}$; there seems to be a contradiction.

This paradox can be resolved by noting that the electromagnet: field between the two particles has associated with it some momentum, and the rate of change of this momentum must be included in the force required to accelerate our system. We could calculate the momentum in the field by evaluating the Poynting vector $\underline{N} = \underline{E} \times \underline{B}$ and integrating over all space. (Note that in this calculation we must exclude the contribution from electric and magnetic fields from the same particle, since this is already included in the properties of the single particle on its own and it contributes to the rest mass m_o). We should expect the result that the momentum associated with the field is just $e V / c^2$, in order to account for the paradox above.

There is an easier route to evaluate the momentum associated with the field, however. In units where $c = 1$, imagine that the two particles are distance x apart and that they are being accelerated with acceleration a in a direction perpendicular to the line joining the two particles. In this situation the electric force between the particles has a component in a direction opposed to the acceleration; this arises because it is necessary to use the retarded potential in evaluating the forces. In the time x for light to travel between the particles, each particle accelerates through an extra distance $\frac{1}{2} a x^2$ to first order. The electric field therefore acts in a direction inclined at angle θ to the line between the particles, where $\theta = (\frac{1}{2} a x^2) / x = \frac{1}{2} a x$. The component of the force opposed to the acceleration is therefore, per particle:

$$F = - e^2 / (4 \pi \epsilon_0 x^2) \cdot \theta \tag{6.8}$$

$$= \frac{1}{2} e V a$$

If this equation is integrated up, and account is taken of the forces on both particles, we deduce that the momentum associated with the field between the particles is $e V \underline{u}$, as we required in order to clear up our paradox.

To conclude this section, there are three answers to the question, 'what is the mass of a particle which is in a potential field ? ' If the system of charges which generates the potential field is fixed, then the rest mass m_o of the particle should be used. If the system of charges is separately caused to move at the same velocity as the particle, and we are interested in the quantization of the particle, then the corrected mass $m = m_o + eV/c^2$ should be used; in this case caution should be employed in order to avoid double-counting of the electromagnetic fields of the system. And if we are interested in the quantization of a complete set of charged particles, then the gravitational mass $m = E/c^2$ should be used, where E is the relativistic energy of the system.

7 Gauge transformations

The electromagnetic gauge transformation has many similarities with the Galilean transformation. The formalisms describing the two can be cast into similar terms. In this section we describe this formalism. Further analysis of the gauge transformation is given in appendix 1.

We shall start with the simple state-vector:

$$|\psi\rangle \quad = \quad a^{+}_{k1}\; a^{+}_{k2} \cdots \; a^{+}_{kN}\; |0\rangle \tag{7.1}$$

This state-vector contains N particles in the exact Fourier states k1 through to kN , and it is clear that the general N-particle state-vector may be constructed from a sum of states of the form (7.1). We shall define our new operator G(A) so that the effect of G(A) upon the simple state-vector above is:

$$G(A)\,|\psi\rangle \quad = \quad a^{+}_{k1+eA} \;\cdots\; a^{+}_{kN+eA}\; |0\rangle \tag{7.2}$$

where e is a **scalar** which depends upon the type of particle involved.

From (2.4) one can deduce the form of the full unobservable transformation based upon $G(A)$. It consists of a transformation of the state-vectors as given by $|\psi\rangle \rightarrow G(A)\,|\psi\rangle$ together with the corresponding transformation of the operators, given by:

$$\hat{O}_H \longrightarrow G(A)\;\hat{O}_H\;G^{+}(A) \tag{7.3}$$

The result that this full transformation can never be observed follows directly from the general arguments which were given in section 2.

In order to see more fully the significance of this transformation, we try to find an alternative form for the transformation of operators (7.3). Consider the momentum operator $\hat{P}$, which was defined in (3.3). It will be clear from (7.1) and (7.2) that the effect of $G(A)$ is to increase the total momentum of a state by an amount $\hat{N}eA$, where $\hat{N}$ is the operator for the total number of particles. Hence it follows that:

$$[\hat{P}, G(A)] = \hat{N}eA\, G(A) \tag{7.4}$$

In order to see how the operator for total momentum $\hat{P}$ is transformed under the $G(A)$, we can use this property (7.4) together with (7.3) with $\hat{O}_H$ replaced by $\hat{P}$. The transformation of $\hat{P}$ is now:

$$\hat{P} \longrightarrow \hat{P} - \hat{N}eA \tag{7.5}$$

It will be clear that the operator for position $\hat{X}$ is unchanged under the transformation, since $\hat{X}$ commutes with the $G(A)$ operator on account of the fact that $G(A)$ does not affect the positions of any of the particles in a state-vector. Similarly, other operators which do not depend upon the fourier-transform space configuration of a system (and hence which do not depend upon the operator P) will be unaffected by the transformation. Hence we can replace (7.3) by (7.5) in general, thus giving a more informative description of the transformation.

We shall call the combined transformation (7.2) and (7.5) the 'gauge' transformation. The interpretation of this transformation is that there is some sort of field, A, which in our unobservable transformation is independent of position; the application of the gauge transformation to the state-vector causes the field to act upon the particles in the system, whereas the application of the transformation to the operators causes the field to act upon any detectors in the system. Any changes in the observables due to the first process are exactly cancelled by the changes to the detectors in the second. This interpretation is analagous to that given for the spatial translation operator in section 3, and for the Galilean transformation in section 4.

How can the gauge transformation be generalized from a hypothetical, unobservable and hence useless formalism to something a little more informative ? There are two ways in which this can be done. Imagine that we have a particle which is at rest in a region of space, and which is being observed from a different region. If an A field is applied to the particle's region of space, but not to the space belonging to the detector, then we would expect that there may be some measureable effect. Indeed, if there were no effect upon the particle then the detector would see the total momentum of the particle increase by an amount eA ; and if we suppose that momentum is conserved in such situations then we deduce that the particle must start to move in order to keep its momentum constant. In this case it is the time derivative of the A field along the path of the particle, DA/Dt , which is the observable. (In terms of electromagnetic theory, this is related to the electric field seen by the particle). The second way in which the gauge transformation can be generalized is by imagining that the field A varies with position in space, so that the curl of the field is not zero. To see how such a field could be observable, imagine a particle which is in a quantized state of zero angular momentum, i.e. the integral of the local momentum of the particle around any loop is zero. If an A field with a non-zero curl is applied to this system, then if there were no further effect, the integral of the local momentum of the particle around any loop would be given by the integral $\int e \text{ curl } \underline{A} \; d\underline{s}$, where $\underline{s}$ is the area of the loop. Since this is not zero, the quantization of angular momentum no longer holds in this situation; and in order to make good this quantization again there must be some (observable) further changes to the state-vector for the particle. We can therefore deduce that the curl of an A field is observable. (In electromagnetic theory this is related to a magnetic field. We shall discuss magnetic fields in these terms later in this chapter).

In this section we have inferred the possible existence of fields which behave similarly to electric and magnetic fields in their effects upon particles. It is possible to infer from symmetry arguments of the above type that these fields obey the Maxwell equations, and this process is described in appendix 1. We shall henceforth assume that these fields may be identified with those of electromagnetism. We shall talk of the 'charge of a particle' to refer to our quantity e , and the 'magnetic vector potential' to refer to our A field.

We finish this section on a philosophical note. Does the electromagnetic vector potential $\underline{A}$ have any "real" existence, or is it merely a mathematical construct with no physical significance ? Derivatives of the $\underline{A}$ field may be measured, but one can never know the absolute value of the vector potential. This observation can be used to imply that $\underline{A}$ is merely a useful mathematical symbol and has no further meaning "behind" physics.

If one subscribes to this view, then for consistency one should also regard the idea of velocity in the same light. One can measure derivatives of velocity (rotation, acceleration) but can never know one's absolute velocity in space. The $\underline{u}$ and $\underline{A}$ fields have precisely analagous roles in physics.

8 Magnetic and rotation fields

In this section we try to bring out the analogy between magnetic and rotation fields, laying the groundwork for later chapters. We shall be particularly concerned with combinations of rotation and magnetic field which leave the quantum state of a system unchanged, at least to some approximation. Later in this section we speculate on the role of Mach's principle with regard to this formalism.

We have seen in section 7 how it occurs that the curl of a magnetic vector potential is observable. There we considered a particle which has charge e and which is in a quantized state of zero angular momentum; i.e. the integral of its canonical momentum around any loop is zero, which condition we shall write as curl $\underline{p} = 0$. We saw how the application of an $\underline{A}$ field which has curl $\underline{A} = \underline{B}$ causes the momentum of the particle to obey:

$$\text{curl } \underline{p} = e\underline{B} \tag{8.1}$$

The quantization condition is therefore no longer obeyed and so there must be further changes to the wavefunction for the particle to make good the quantization; it is these changes which render observable the $\underline{B}$ field.

A similar analysis may be made for a rotation field $\underline{\omega}$. If a system is set rotating with angular velocity $\underline{\omega}$ so that the local velocity is $\underline{u} = \underline{\omega} \times \underline{r}$, then the local velocity obeys curl $\underline{u} = 2\,\underline{\omega}$. The application of a rotation field to our system there causes the particle to have momentum given by:

$$\text{curl } \underline{p} = 2m\omega \tag{8.2}$$

Again the quantization of the particle causes further changes to the wavefunction, thereby rendering observable the rotation field

From this analysis, it follows that a rotation can be observed intrinsically by our system; that is, it is possible to tell that one is rotating, without reference to bodies outside one's experiment. There must therefore be some reference frame of non-rotation which is referred to in some sense by our experiment in order to determine the rotation velocity. It was Mach who originally asked questions as to the[1] nature of this reference frame of non-rotation: he suggested that all rotation velocities, like all spatial velocities, should be measured relative to something, rather than having an absolute existence on their own. In other words, he suggested that the frame of non-rotation is not a property of space itself, but is associated with the existence of some 'rotation field', through which the rotation velocity of one's experiment can be compared to the rotation velocity of (say) the rest of the matter in universe. This could have a number of consequences. If all the rest of the matter in the universe could be eliminated then our little experiment would have no reference frame of non-rotation to which it could refer, and so it would be unable to tell its rotation velocity. More significantly for feasible experiments, it might be that the rotation field had a spatial dependence so that the non-rotating frame at one point in space might differ from the non-rotating frame at some other point in space.

If Mach's suppositions are to be believed, then it would seem reasonable to ask the same questions of the frame of zero magnetic field. From (7.1) and (7.2) it is clear that a magnetic field and a rotation field have precisely analagous effects upon the matter in the universe; therefore it would seem reasonable to ask whether all magnetic fields should be measured relative to something, as we have asked of rotation fields. If all the matter in the universe were to be eliminated, would we still be able to measure a magnetic field ? Is there some field similar to that rotation field which is supposed to emanate from the matter in the universe, through which magnetic fields are compared to zero ? These questions, like those originally asked by Mach, remain unanswered.

For a system of particles with a constant ratio of charge to mass, a particular combination of a magnetic field and a rotation field is *not* observable. From (8.1) and (8.2), if a rotation field $\underline{\omega}$ and a magnetic field $\underline{B}$ are applied to our system, and if their ratio is:

$$e\underline{B} = -2m\omega \tag{8.3}$$

then the momentum of our particles $\underline{p}$ is nowhere changed and so the quantization condition, that curl $\underline{p} = 0$ for a particle in a quantum state of zero angular momentum, is still obeyed in the new system. We therefore expect that this transformation is not observable.

This equation (8.3) has significance for the[4] theory of superconductivity. The electrons in a superconductor are associated with a parameter which has macroscopic quantum-mechanical properties (the order parameter); and in particular this parameter is ordinarily in a state of zero angular momentum. A sample of superconductor which is not rotating exhibits the 'Meissner effect', i.e. all magnetic fields are expelled from the bulk of the superconductor. If the combined transformation is made to a rotation and a magnetic field, with the equality (8.3), the quantization of the order parameter is unaffected and so the superconductor is still in a valid quantum-mechanical state. It turns out that the superconductor now exhibits a Meissner-like effect in which this field $\underline{B}$ is maintained in the bulk of the rotating superconductor. We shall discuss this in detail later.

Thus far in our discussion of this combined transformation we have used the Galilean transformation. In other words, we have ignored effects which are not first order in the velocity : we have assumed that the Lorentz parameter γ is just one, and we have ignored centrifugal effects. We now try to give a more sophisticated analysis which includes relativistic effects in the rotation velocity, and we shall see that the transformation (8.3) is not unobservable to order u^2. We start by analysing centrifugal (i.e. $O(u^2)$) effects.

We shall consider an infinitely long cylinder which contains a uniform density of charged particles, each of which is in a state of zero angular momentum. This idealized system is particularly easy to analyze, and it displays some of the properties we require. Imagine that this cylinder is set rotating at angular velocity $\underline{\omega}$ about its axis, and a uniform magnetic field $\underline{B}$ applied along the axis has the magnitude given by (8.3). The quantization condition of the particles is still met, but are there any radial forces acting upon the particles

There are three contributions to the radial force upon each particle. The centrifugal force, which is outwards, is of magnitude $m\, r\, \omega^2$. The force due to the motion of the particles through the magnetic field is inwards, and of magnitude $e\, B\, r\, \omega \;=\; 2\, m\, r\, \omega^2$. The sum of these two is a net radial inwards force $m\, r\, \omega^2$. However, there is also the repulsion of the charged particles to take into account. If the charge density is $e\, \rho$, then the electric field due to the particles is radially outwards, and of magnitude $E \;= \rho\, e\, r\, /\, (\, 2\, \epsilon_o\,)$; the outwards force is therefore $e\, E = e^2 \rho\, r\, /\, (\, 2\, \epsilon_o\,)$. If the charge density can be arranged so that:

$$\rho \quad = \quad 2\, \epsilon_0\, m\, \omega^2 / e^2 \tag{8.4}$$

then there is no net force upon the particles, and the situation is stable. There will of course be a correction to the magnetic field in some cases, to allow for the field generated by the rotating charges; this correction has so far been ignored.

We shall consider a system where the charge density is given by (8.4), and the geometry of the cylinder is such that the magnetic field due to the rotating charges themsleves can be ignored. We have shown that the quantization of the particles is satisfied, and also that there are no forces acting upon the particles in the rotating frame. As a check upon the unobservability of the rotation, we now verify that there are no unexpected forces upon particles moving relative to the rotating frame. If a particle is moving radially outwards with velocity v_r, then the forces upon it are $e\,v_r\,B = 2\,v_r\,m\,\omega$, from the magnetic field, and the negative of this from the Coriolis field. These therefore cancel. If the particle is moving at velocity v_t tangentially relative to the rotating frame, the forces are $e\,B\,(v_t + r\,\omega) = 2\,m\,r\,\omega^2 + 2\,m\,r\,\omega\,v_t$, on account of the magnetic field; $m(\,r\omega + v_t)^2/r - m\,v_t^2\,/\,r = m\,r\,\omega^2 + 2\,m\,r\,\omega\,v_t$ from the centrifugal force, and $m\,r\,\omega^2$ from the electric field. The sum of these three is zero, as we expect.

We can therefore conclude that for non-relativistic rotation rates and for a long cylindrical geometry, the transformation to a frame carrying a B field and the introduction of a constant charge density to the system, these given by (8.4) and (8.3), is an unobservable transformation for particles of constant ratio of charge to mass. If the particles are in a good quantum state before the transformation, they remain so after the transformation has been carried out. All particle trajectories in the new frame are the analogues of what they would have been had the transformation not been carried out.

It is of interest to check that there is no preferred axis of rotation in this system - i.e. an observer anywhere in the system could analyze the motion as if he were at the centre of rotation, and he would obtain the same answers as any other observer. This is easily checked by writing the local velocity as $\underline{u} = \underline{\omega} \times \underline{r}$. An observer at position $\underline{a}$ would analyze the motion as if it were described by $\underline{u}' = \underline{\omega} \times (\,\underline{r} + \underline{a}\,) = \underline{u} + \underline{u}_o$, where $\underline{u}_o = \underline{\omega} \times \underline{a}$ is a constant velocity. The two observers are related by a velocity transformation, and so they do indeed both obtain the correct answers as far as the velocities are concerned. A similar analysis holds for the $\underline{A}$ field; the two observers are related by a gauge transformation. The charge density is a constant in space and so looks the same to all observers. The centrifugal acceleration is suffered by particles and observers alike, so the observer cannot tell. There is therefore no preferred axis of rotation.

There is some difficulty in extending these ideas to all orders in the velocity. What is meant by a 'rigid rotation' when the velocity of rotation at the edge of the system becomes close to that to light ? If one naively imagines that the motion corresponds to classical ideas of rigid rotation, then there is a contradiction at the point where the tangential velocity would equal that of light.

We shall make a different definition of what we mean by rotation in the case of high tangential velocity. We shall assume that there is no preferred axis of rotation. If a transformation is made so that the velocity and A field at the point $\underline{O}$ are both equal to zero, then we define that, locally:

$$\begin{aligned} \mathrm{curl}\,(\gamma m \underline{u}) &= 2\underline{\omega} \\ &= -e\,\mathrm{curl}\,\underline{A} \end{aligned} \tag{8.5}$$

Since all paths are locally the same as their counterparts before the transformation to the rotating frame, then globally the whole transformation is not observable. (We have not considered here the question of how the $\underline{A}$ field is maintained in the face of the moving charges which oppose the field). This transformation has a number of interesting properties. For example, from the definition (8.5) the velocity at radius 2r from our (arbitrary) origin is the relativistic sum of twice the velocity at radius r , i.e.

$$u(2r) = 2\,u(r) / (1 + u(r)^2) \tag{8.6}$$

which has the solution that:

$$u(r) = \tanh\,(r/r_0) \tag{8.7}$$

where r_o is a constant.

In the case of our particularly simple system, in which all particles have the same ratio of charge to mass and there is a simple cylindrical geometry, we have now come some way towards an answer to Mach's questions. In our system there is no absolute frame of non-rotation and there is no absolute frame of zero magnetic field; the two are related. We can perform a transformation from one frame to another in which the magnetic field and the rotation velocities are different, and yet an observer who uses only the charged particles as detectors can never tell the difference. The transformation remains unobservable to all orders in the fields, i.e. even highly relativistic velocities can be incorporated in the theory.

Our system clearly displays many of the properties which were desired by Mach. Yet in our universe, it is not true that all particles possess the same ratio of charge to mass, and so this system cannot form the basis of some model of the universe.

It is interesting to note that an assumption of general relativity is that the ratio of gravitational charge to mass is a constant for all particles in the universe - i.e. the interaction of all particles with the gravitational field is proportional to the mass of the particle. Perhaps a set of transformations based upon the gravitational rather than electromagnetic field could have the properties for which Mach had hoped.

In the theory of general relativity there is indeed an analogue of the magnetic field, and it is[5,6] called the Lenz-Thirring effect. A sensitive gyroscope placed in the vicinity of a massive rotating body will be caused to precess on account of this field, just as if the local frame of non-rotation were perturbed by the presence of the massive rotating body. The magnitude of this effect at the Earth's surface due to the rotation of the earth and to its motion around the sun is of order 10^{-15} radians per second, a figure so small that the effect has not been measured successfully to date.

Yet there are further problems in applying the Lenz-Thirring field together with a rotation in order to generate an unobservable transformation. In the case of electromagnetism, we found that there was a force towards the central axis of rotation when this combined transformation was made. There we could null out this centre-seeking force by introducing a constant charge density. But in the case of gravitation there is no way in which particles can be made to repel one another - the gravitational force is always attractive. In other words, if we assume that there is a direct analogy[7] between the gravitational and electromagnetic fields, it becomes impossible to generate an unobservable transformation which involves a rotation and a nulling gravitational field. If the analogy were valid, then we could say that the field which Mach assumed to emanate from the fixed stars and which was supposed to fix our local frame of non-rotation, could not be gravitational in origin. If it were gravitational in origin then we would observe that all objects would attract one another in a certain plane, but that they would not attract in a direction normal to the plane.

There is scope for further work on this topic.

Appendix 1 - Gauge Theory in Particle Physics

Gauge theory is usually considered the realm of the particle physicist. We quote in this appendix an extract from a series of lectures on particle physics, which show the power of this approach. Much of the work of this chapter is directed towards translating the gauge theory to the notation of solid state physics.

Only small notational changes have been made to the text of the extract in order to conform with the treatment to be given later. The standard notation of Dirac theory[2] is used: it should be noted that in this notation the symbol $\not\partial$ means the contraction $\gamma^\mu \partial_\mu$ where γ^μ are the usual four Dirac matrices and the quantity ∂_μ signifies the derivative $\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ in units $c = 1$.

" At the basis of every symmetry principle in physics there is an assumption that some quantity is not measureable. For example, the assumption that there is no absolute position in space leads to the invariance under translations. Here we shall be interested internal symmetries, i.e. transformations which do not affect the space-time point x. A simple example is given by the Lagrangian density of a free fermion field $\psi(x)$:

$$L_o = \bar{\psi}(x) \ (i \not{\partial} - m) \ \psi(x) \tag{1}$$

which is invariant under the phase transformation

$$\psi(x) \longrightarrow e^{i\theta} \psi(x) \tag{2a}$$

$$\partial_\mu \psi(x) \longrightarrow e^{i\theta} \partial_\mu \psi(x) \tag{2b}$$

where θ is an arbitrary, x-independent phase. Formula (2b) follows from (2a), i.e. the derivative of the field transforms like the field itself.

. . . .

" The invariance of L_o under (2) means that the phase of the field $\psi(x)$ is not measurable, therefore it can be chosen arbitrarily. On the other hand, since it is x-independent, it must be chosen to be the same over the entire universe for all times. This situation is clearly unsatisfactory on physical grounds. We would like instead to have a formalism which would allow us to fix the phase locally in a region with the dimensions of our experiment without reference to far-away distances; in other words we would like to replace (2a) by:

$$\psi(x) \longrightarrow e^{i\theta(x)} \psi(x) \tag{5a}$$

where θ is now a function of x. I want to emphasize here that this requirement is based on purely aesthetic arguments. If we adopt (5a) as a symmetry transformation, we face a serious problem, because now (2b) is replaced by:

$$\partial_\mu \psi(x) \longrightarrow e^{i\theta(x)} \, \partial_\mu \psi(x) + i\, e^{i\theta(x)} \, \psi(x) \, \partial_\mu \theta(x) \tag{5b}$$

i.e. the derivative of the field no longer transforms like the field itself and, as a result, the Lagrangian (1) is no longer invariant under (5). We shall call transformations of the form (5), i.e. with x-dependent parameters, "local" or "gauge" transformations.

" In differential geometry there is a standard way of restoring invariance under (5). Since the trouble arises from the derivative operator, we must introduce a new "derivative" D , called the "covariant derivative", which is again a first order differential operator, but with the property that it transforms under (5a) like the field itself:

$$D_\mu \psi(x) \longrightarrow e^{i\theta(x)} \, D_\mu \psi(x) \tag{5c}$$

In order to find such a D_μ , we first introduce the affine connection which, in our language, is related to the "gauge field" $A_\mu(x)$ and which, by definition, transforms like:

$$A_\mu(x) \longrightarrow A_\mu(x) + (1/e) \, \partial_\mu \theta(x) \tag{6}$$

with e a constant. We then define D_μ by

$$D_\mu \equiv \partial_\mu + i\, e\, A_\mu \tag{7}$$

and it is easy to verify that $D_\mu \psi(x)$ does transform under (5a) and (6), as does (5c). Invariance under gauge transformations is now restored by replacing ∂_μ by D_μ everywhere in L_o.

$$\begin{aligned} L_o \rightarrow L_1 &\equiv \bar{\psi}(x) \, (i \not{D} - m) \, \psi(x) \\ &= \bar{\psi}(x) \, (i \not{\partial} - m) \, \psi(x) - e \, \bar{\psi}(x) \, \gamma_\mu \psi(x) \, . \, A^\mu(x) \end{aligned} \tag{8}$$

The Lagrangian (8) is invariant under (5a) and (6), and it contains the gauge field $A_\mu(x)$. If we want to interpret the latter as the field representing the photon, we must add to (8) a term corresponding to its kinetic energy. This term must be, by itself, gauge-invariant, and we are thus easily led to the final Lagrangian

$$L_1 \rightarrow L_2 \;\; \overset{=}{-} \;\; L_1 - \frac{1}{4} F^{\mu\nu}(x) \, F_{\mu\nu}(x) \tag{9}$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{10}$$

Equation (9) is nothing else but the familiar Lagrangian of quantum electrodynamics. We have obtained it just by imposing invariance under gauge transformations. A final remark is in order: L_2 does not contain a term proportional to $A_\mu A^\mu$, since this term is not invariant under (6). In other words, gauge invariance forces the photon to be massless."

Extracted from "An introduction to gauge theories"
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