

Exact quantization of multistage stochastic linear problems

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Abstract

We show that the multistage linear problem (MSLP) with an arbitrary cost distribution is equivalent to a MSLP on a finite scenario tree. We establish this exact quantization result by analyzing the polyhedral structure of MSLPs. In particular, we show that the expected cost-to-go functions are polyhedral and affine on the cells of a chamber complex, which is independent of the cost distribution. This leads to new complexity results, showing that MSLP is fixed-parameter tractable.

1 Introduction

Stochastic programming is a powerful modeling paradigm for optimization under uncertainty that has found many applications in energy, logistics or finance (see e.g. [WZ05]). Multistage linear stochastic programs (MLSP) constitute an important class of stochastic programs. They have been thoroughly studied, see e.g. [BL11, Pré13]. One reason for this interest is the availability of efficient linear solvers and the use of dedicated algorithms leveraging the special structure of linear stochastic programs ([VSW69, Bir85]).

In this paper, we show that every MSLP with general cost distribution is equivalent to an MSLP with finite distribution. This leads to explicit representations of their value functions and to new complexity results.

1.1 Multistage stochastic linear programming

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Given a sequence of independent random variables $\mathbf{c}_t \in L_1(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^{n_t})$ and $\boldsymbol{\xi}_t = (\mathbf{A}_t, \mathbf{B}_t, \mathbf{h}_t)$, indexed by $t \in [T] := \{1, \dots, T\}$, we consider the MSLP

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given by

$$\begin{aligned}
\min_{(\mathbf{x}_t)_{t \in [T]}} \quad & c_1^\top x_1 + \mathbb{E} \left[\sum_{t=2}^T c_t^\top \mathbf{x}_t \right] \\
\text{s.t.} \quad & A_1 x_1 \leq b_1 \\
& \mathbf{A}_t \mathbf{x}_t + \mathbf{B}_t \mathbf{x}_{t-1} \leq \mathbf{b}_t \quad \text{a.s.} \quad \forall t \in \{2, \dots, T\} \\
& \mathbf{x}_t \in L_\infty(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^{n_t}) \quad \forall t \in \{2, \dots, T\} \\
& \mathbf{x}_t \preceq \mathcal{F}_t \quad \forall t \in \{2, \dots, T\}
\end{aligned} \tag{MSLP}$$

where $\mathbf{x}_1 \equiv x_1$, $\mathbf{A}_1 \equiv A_1$ and $\mathbf{b}_1 \equiv b_1$ are deterministic and \mathcal{F}_t is the σ -algebra generated by $(\mathbf{c}_2, \boldsymbol{\xi}_2, \dots, \mathbf{c}_t, \boldsymbol{\xi}_t)$. The last constraint, known as non-anticipativity, means that \mathbf{x}_t is measurable with respect to \mathcal{F}_t .

Most results for MSLP with continuous distributions rely on discretizing the distributions. The Sample Average Approximation (SAA) method (see e.g. [SDR14, Chap. 5]) samples the costs and constraints. It relies on probabilistic results based on a uniform law of large number to give statistical guarantees. Obtaining a good approximation requires a large number of scenarios. In order to alleviate the computations, we can use scenario reduction techniques (see [DGKR03, HR03]). Latin Hypercube Sampling and variance reduction methods are also used to produce scenarios. Finally, one generates heuristically “good” scenarios, representing the underlying distribution (see [KW07]). Alternatively, we can leverage the structure of the problem to produce finite scenario trees (see [Kuh06, MAB14, MP18]) that yields bounds for the value of the true optimization problem. In each of these approaches, one solves an approximate version of the stochastic program, with or without statistical guarantee.

1.2 The exact quantization problem

Here, we aim at solving exactly the original problem, by finding an equivalent formulation with discrete distributions. This notion of equivalent formulation is best understood through the dynamic programming approach of MSLP. We define the *cost-to-go* function V_t inductively as follows. We set $V_{T+1} \equiv 0$ and for all $t \in \{2, \dots, T\}$:

$$\begin{aligned}
V_t(x_{t-1}) &:= \mathbb{E} [\hat{V}_t(x_{t-1}, \mathbf{c}_t, \boldsymbol{\xi}_t)] \\
\hat{V}_t(x_{t-1}, c_t, \xi_t) &:= \min_{x_t \in \mathbb{R}^{n_t}} c_t^\top x_t + V_{t+1}(x_t) \\
\text{s.t.} \quad & A_t x_t + B_t x_{t-1} \leq h_t
\end{aligned} \tag{1}$$

where $x_{t-1} \in \mathbb{R}^{n_{t-1}}$, $c_t \in \mathbb{R}^{n_t}$ and $\xi_t = (A_t, B_t, b_t) \in \mathbb{R}^{q_t \times n_{t-1}} \times \mathbb{R}^{q_t \times n_t} \times \mathbb{R}^{q_t}$.

We choose to distinguish the random cost \mathbf{c}_t from the noise $\boldsymbol{\xi}_t$ affecting the constraints. Indeed our results require $\boldsymbol{\xi}_t$ to be finitely supported (see Examples 1 and 2) while \mathbf{c}_t can have a continuous distribution. This separation does not preclude correlation between \mathbf{c}_t and $\boldsymbol{\xi}_t$. However, we require $\{(\mathbf{c}_t, \boldsymbol{\xi}_t)\}_{t \in [T]}$ to be a sequence of independent random variables to leverage Dynamic Programming, even though some results can be extended to dependent $(\boldsymbol{\xi}_t)_{t \in [T]}$.

We say that an MSLP admits an *exact quantization* if there exists a finitely supported $(\check{\mathbf{c}}_t, \check{\boldsymbol{\xi}}_t)_{t \in [T]}$ that yields the same expected cost-to-go functions, $(V_t)_{t \in [T]}$. In particular the MSLP is equivalent to a problem on a finite scenario tree.

An obvious necessary condition for exact quantization is that the value function V_t be a polyhedral function, meaning that it takes value in $\mathbb{R} \cup \{+\infty\}$ and its epigraph is a (possibly empty) polyhedron. Indeed, for each $(c, \xi) \in \text{supp}(\mathbf{c}, \boldsymbol{\xi})$, $Q^{c, \xi} : (x, y) \rightarrow c^\top y + \mathbb{I}_{A_x + B_y \leq h}$ is polyhedral.

Thus, $\hat{V}(\cdot, c, \xi) := \min_{y \in \mathbb{R}^m} Q^{c, \xi}(\cdot, y)$ is polyhedral as $\text{epi } \hat{V}(\cdot, c, \xi)$ is a projection of $\text{epi } Q^{c, \xi}$ (see [JKM08, SDR14]). Hence, the following examples show that if the constraints have non-discrete distributions, there is no hope to have an exact quantization theorem. We shall see, however, that this is the case without restrictions on the cost distribution.

Example 1 (Stochastic \mathbf{B}). Here, and in the next example, \mathbf{u} denotes a uniform random variable on $[0, 1]$.

$$V(x) = \mathbb{E} \left[\begin{array}{l} \min_{y \in \mathbb{R}^m} y \\ \text{s.t. } \mathbf{u}x \leq y \\ 1 \leq y \end{array} \right] = \mathbb{E} [\max(\mathbf{u}x, 1)] = \begin{cases} 1 & \text{if } x \leq 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geq 1 \end{cases}$$

Example 2 (Stochastic \mathbf{b}).

$$V(x) = \mathbb{E} \left[\begin{array}{l} \min_{y \in \mathbb{R}^m} y \\ \text{s.t. } \mathbf{u} \leq y \\ x \leq y \end{array} \right] = \mathbb{E} [\max(x, \mathbf{u})] = \begin{cases} \frac{1}{2} & \text{if } x \leq 0 \\ \frac{x^2+1}{2} & \text{if } x \in [0, 1] \\ x & \text{if } x \geq 1 \end{cases}$$

1.3 Contribution

We rely on a geometric approach, which enlightens the polyhedral structure of MSLP. We first establish exact quantization results in the two-stage case showing that there exists an optimal recourse affine on each cell of a polyhedral complex which is precisely the chamber complex [BS92, RZ96], a fundamental object in combinatorial geometry. A chamber complex is defined as the common refinement of the projections of faces of a polyhedron. In particular, Theorem 9 provides an explicit exact quantization, in which the quantized probabilities and costs are attached to the cones of a polyhedral fan \mathcal{N} (we refer the reader to [DLRS10, Zie12, Grü13, Fuk16] for background on polyhedral complexes and fans). On each cone $N \in \mathcal{N}$, we replace the distribution of $\mathbf{c} \mathbf{1}_{\text{ri} N}$ by a Dirac distribution concentrated on the expected value $\check{c}_N = \mathbb{E}[\mathbf{c} | \mathbf{c} \in \text{ri} N]$, and an associated weight $\check{p}_N = \mathbb{P}[\mathbf{c} \in \text{ri} N]$. Further, \mathcal{N} is *universal* in the sense that it does not depend on the distribution of \mathbf{c} .

In order to extend this result to the multistage case we establish in Lemma 15 a Dynamic Programming type equation in the space of polyhedral complexes. Then we show an exact quantization result in Theorem 18. Again, this quantization is *universal* in the cost distribution.

We apply this polyhedral approach to obtain fixed parameters polynomial time complexity results considering both the exact computation problem and the approximation problem. For distributions that are uniform on polytopes or exponential, we show the MSLP can be solved in a time that is polynomial provided that the horizon T and the dimensions n_2, \dots, n_T of the successive recourses are fixed. The proof relies on the theory of linear programming with oracles [GLS12] as well as on upper bound theorems of McMullen [McM70] and Stanley [Sta75] concerning the number of vertices and the size of a triangulation of a polyhedron. We obtain similar results for the approximation problem. Then the distribution cost can be essentially arbitrary: we only assume that it is given implicitly through an appropriate oracle. This applies in particular to distributions with a smooth density with respect to Lebesgue measure.

In summary, our main contributions are the following:

1. MSLP with arbitrary cost distribution and finitely supported constraints admit an exact quantization result, *i.e.* are equivalent to MSLP with discrete cost distribution;

2. the cost-to-go functions of such MSLP are polyhedral and affine on the cells of a universal polyhedral complex (*i.e.* independent of the cost distribution);
3. exact formulas for quantized costs and probabilities in the case of exponentially or uniformly distributed costs on a polytope;
4. fixed-parameter polynomial time tractability results for 2SLP and MSLP.

1.4 Comparison with related work

A combinatorial approach of deterministic parametric linear programming was developed by Walkup and Wets [WW69] see also [ST97] for a more recent discussion. Their basis decomposition theorem describes how the value of a linear program in standard form varies with respect to the cost and to the right-hand side of the constraints. In the two stage case, we can see the collection of rows of A as a vector configuration, and the right-hand side of the recourse problem $b - Bx$ as a height function which determines a regular subdivision of this configuration. The space of regular subdivision is represented by the so called secondary fan [DLRS10]. We may apply this theorem to the dual problem of the recourse problem to deduce that the expected cost-to-go function is affine on each cell of an affine section of the secondary fan. This affine section can be shown to coincide with the chamber complex used here. However, the basis decomposition theorem cannot be applied to the extensive form of a *multistage* problem. In particular nonanticipativity constraints cannot be tackled in this way. Thus, we choose to develop an approach through chamber complexes as it is more direct, allowing us to obtain also a result in the multistage case. The comparison with the approach of Walkup and Wets is further discussed in Section 3.2.

The complexity of stochastic programming has been extensively studied. Dyer and Stougie [DS06] proved that 2 stage stochastic programming is $\sharp P$ -hard in the discrete case, by reducing the problem of graph reliability to the discrete distribution case. They stated that the computation of the volume of a polytope can be reduced to the continuous distribution case, a result which was subsequently proved in [HKW16]. Computing the volume of a polytope, as well as graph reliability, is $\sharp P$ -complete. Hanasusanto, Kuhn and Wiesemann (*ibid*) showed that computing an approximate solution to the 2-stage linear programming (2SLP) with continuous distribution with a sufficiently high accuracy is also $\sharp P$ -hard. Other papers [SN05] studied the complexity of 2-stage linear programming 2SLP and MSLP. Most complexity results there are hardness results. In contrast, we prove that 2SLP and MSLP are fixed parameter tractable.

Finally, Lan [Lan20] and Zhang and Sun [ZS19] independently analysed the complexity of Stochastic Dual Dynamic Programming (SDDP). It follows from their results that finitely supported MSLP can be solved approximately in pseudo-polynomial time in the error approximation ε when all the dimensions and the horizon are fixed. In other words the complexity of these SDDP methods is polynomially bounded in $1/\varepsilon$. In contrast, our approach shows that MSLP can be solved approximately in polynomial time in $\log(1/\varepsilon)$, when T, n_2, \dots, n_T are fixed. In particular, the first state dimension is not fixed. Moreover, we obtain polynomial complexity bounds in the exact (Turing) model of computation for appropriate classes of distributions. Note that in the approach presented here, contrary to SDDP like methods, we do not rely on statistical sampling and the value functions are computed exactly in one pass only. However, the objective of SDDP is to obtain quickly an approximate solution whereas our approach computes exactly all the supporting hyperplanes.

1.5 Structure of the paper

We recall, in Section 2, notions from the theory of polyhedra: *polyhedral complexes*, *normal fans* and *chamber complexes*. In Section 3 we establish the exact quantization result for 2-stage stochastic linear programming. In Section 4, we show that chamber complexes can be propagated through dynamic programming, leading to the exact quantization result for the MSLP. We show in Section 5 how the quantized probabilities and cost can be computed for appropriate distributions. Finally, in Section 6, we draw the consequences of our results in terms of computational complexity.

1.6 Notation

As a general guideline **bold** letters denote random variables, normal scripts their realisation. Capital letters denote matrices or sets, calligraphic (e.g. \mathcal{N}) denote collections of sets. The indicator function $\mathbb{1}_P$ (resp. $\mathbf{1}_P$) takes value 0 (resp. 1) if P is true and $+\infty$ (resp. 0) otherwise. We set $[k] := \{1, \dots, k\}$, and we denote by $\#E$ the cardinal of a set E . We denote by $\text{Cone}(A) := A\mathbb{R}_+^n$ the cone hull of the columns of A . $x \leq y$ is the standard partial order, given by $\forall i, x_i \leq y_i$. $F \triangleleft G$ if F is a subface of G . $\mathcal{P} \preceq \mathcal{Q}$ if \mathcal{P} is a refinement of the polyhedral complex \mathcal{Q} . $\text{supp } \mathcal{C} := \bigcup_{C \in \mathcal{C}} C$ is the support of a collection of sets \mathcal{C} , \mathcal{C}^{\max} : the sets of maximal elements of a collection of sets \mathcal{C} . $\text{rc}(P)$ is the recession cone of a polyhedron P . For a polyhedron P , we denote $\mathcal{F}(P)$ its faces, $\text{Vert}(P)$ its vertices and $\text{Ray}(P)$ a set with vectors each representing one extreme rays (for example the normalized extreme rays). P^ψ is the face of P given by $\arg \min_{x \in P} \psi^\top x$. $N_P(x)$ is the normal cone of P at x , and $\mathcal{N}(P)$ the normal fan of P .

2 Polyhedral tools

Our proofs rely on the notions of normal fan and chamber complex of a polyhedron recalled here. These polyhedral objects reveal the geometrical structure of MSLP. Both the normal fan and the chamber complex are special polyhedral complexes.

2.1 Polyhedral complexes

Polyhedral complexes are collections of polyhedra satisfying some combinatorial and geometrical properties. In particular the relative interiors of the elements of a polyhedral complex (without the empty set) form a partition of their union. We refer to [DLRS10] for a complete introduction to polyhedral complexes and triangulations.

Definition 1 (Polyhedral complex). *A finite collection of polyhedra \mathcal{C} is a polyhedral complex if it satisfies i) if $P \in \mathcal{C}$ and F is a non-empty¹ face of P then $F \in \mathcal{C}$ and ii) if P and Q are in \mathcal{C} , then $P \cap Q$ is a (possibly empty) face of P .*

We denote by $\text{supp } \mathcal{C} := \bigcup_{P \in \mathcal{C}} P$ the support of a polyhedral complex. Further, if all the elements of \mathcal{C} are polytopes (resp. cones, simplices, simplicial cones), we say that \mathcal{C} is a polytopal complex (resp. a fan, a simplicial complex, a simplicial fan).

We recall that a *simplex* of dimension d is the convex hull of $d + 1$ affinely independent point and that a *simplicial cone* of dimension d is the conical hull of d linearly independent vectors.

¹For some authors, a polyhedral complex must contain the empty set. We do not make this requirement.

Proposition 2. For any polyhedral complex \mathcal{C} , the relative interiors of its elements (without the empty set) form a partition of its support: $\text{supp}(\mathcal{C}) = \bigsqcup_{P \in \mathcal{C}} \text{ri}(P)$.

For example, the set of faces $\mathcal{F}(P)$ of a polyhedron P is a polyhedral complex.

Definition 3 (Refinements and triangulation). Let \mathcal{C} and \mathcal{R} be two polyhedral complexes, we say that \mathcal{R} is a refinement of \mathcal{C} , denoted $\mathcal{R} \preceq \mathcal{C}$, if $\text{supp} \mathcal{R} = \text{supp} \mathcal{C}$ and for every cell $R \in \mathcal{R}$ there exists a cell $C \in \mathcal{C}$ containing R : $R \subset C$.

Note that \preceq defines a partial order on the space of polyhedral complexes, and the meet associated with this order is given by the common refinement of two polyhedral complexes \mathcal{C} and \mathcal{C}' defined as the polyhedral complex of the intersections of cells of \mathcal{C} and \mathcal{C}' :

$$\mathcal{C} \wedge \mathcal{C}' := \{R \cap R' \mid R \in \mathcal{C}, R' \in \mathcal{C}'\}$$

A triangulation \mathcal{T} of a polytope Q is a refinement of $\mathcal{F}(Q)$ such that the cells of dimension 0 of \mathcal{T} are the vertices of Q and \mathcal{T} is a simplicial complex. A triangulation \mathcal{T} of a cone K is a refinement of $\mathcal{F}(K)$ such that the cells of dimension 1 of \mathcal{T} are the rays of K and \mathcal{T} is a simplicial fan.

2.2 Normal fan

The normal fan is the collection of the normal cones of all faces of a polyhedron. See [LR08] for a review of normal fan properties.

Recall that the *normal cone* of a convex set $C \subset \mathbb{R}^m$ at the point x is the set $N_C(x) := \{\alpha \in \mathbb{R}^m \mid \forall y \in C, \alpha^\top(y - x) \leq 0\}$. More generally, for a set $E \subset C$, $N_C(E) := \bigcap_{x \in E} N_C(x)$.

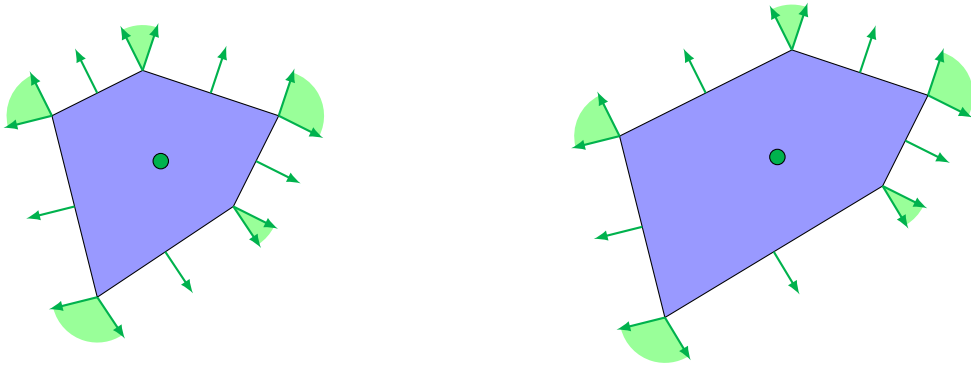


Figure 1: Two normally equivalent polytopes P and P' and their normal fan $\mathcal{N}(P) = \mathcal{N}(P')$.

Definition 4 (Normal Fan). The normal fan² of a convex set C is the collection of polyhedral cones

$$\mathcal{N}(C) := \{N_C(x) \mid x \in C\}$$

We say that two convex sets C and C' are normally equivalent if they have the same normal fan: $\mathcal{N}(C) = \mathcal{N}(C')$.

²Sometimes called *outer* normal cones and fan, as opposed to *inner* cones obtained either by inverting the inequality in the definition of the normal cone or by taking the opposite cones respect to the origin.

Recall that the *polar* of a convex set C is the set $C^\circ := \{\alpha \mid \forall x \in C, \alpha^\top x \leq 0\} = N_C(0)$ and the *recession cone* of a convex set C is given by $\text{rc}(C) := \{r \in C \mid \forall \mu \in \mathbb{R}_+, \forall x \in c, x + \mu r \in C\}$. In particular, for a polyhedron, the recession cone and its polar are given by

$$\text{rc}(\{x \mid Ax \leq b\}) = \{x \mid Ax \leq 0\} \quad \text{rc}(\{x \mid Ax \leq b\})^\circ = \text{Cone}(A^\top). \quad (2)$$

Proposition 5 (Basic properties of normal fans (see e.g. [LR08])). *If P is a polyhedron, the normal fan $\mathcal{N}(P)$ is a finite collection of polyhedral cones (and in particular a polyhedral complex). Further, the support of $\mathcal{N}(P)$ can be expressed geometrically as the polar of the recession cone of P , i.e.*

$$\text{supp } \mathcal{N}(P) = (\text{rc}(P))^\circ \quad (3)$$

2.3 Chamber complex

The affine regions of the cost-to-go function will correspond to cells of a chamber complex. Projections of polyhedra, fibers and chambers complexes are studied in [BS92, RZ96, Ram96].

Definition 6 (Chamber complex). *Let $P \subset \mathbb{R}^n$ be a polyhedron and π a linear projection defined on \mathbb{R}^n . For $x \in \pi(P)$ we define the chamber of x for P along π as*

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F).$$

The chamber complex $\mathcal{C}(P, \pi)$ of P along π is defined as the (finite) collection of chambers, i.e.

$$\mathcal{C}(P, \pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}.$$

Further $\mathcal{C}(P, \pi)$ is a polyhedral complex such that $\text{supp } \mathcal{C}(P, \pi) = \pi(P)$. In particular, $\{\text{ri}(\sigma) \mid \sigma \in \mathcal{C}(P, \pi)\}$ is a partition of $\pi(P)$.

More generally, the chamber complex of a polyhedral complex \mathcal{P} is

$$\mathcal{C}(\mathcal{P}, \pi) := \{\sigma_{\mathcal{P},\pi}(x) \mid x \in \pi(\text{supp}(\mathcal{P}))\}.$$

with $\sigma_{\mathcal{P},\pi}(x) := \bigcap_{F \in \mathcal{P} \text{ s.t. } x \in \pi(F)} \pi(F)$.

Lemma 7 (Chamber complex monotonicity with respect to refinement order). *Consider two polyhedral complexes of \mathbb{R}^d and a projection π . If $\mathcal{R} \preceq \mathcal{S}$ then $\mathcal{C}(\mathcal{R}, \pi) \preceq \mathcal{C}(\mathcal{S}, \pi)$.*

Proof. For any $R \in \mathcal{R}$, there exist $S_R \in \mathcal{S}$ such that $R \subset S_R$. Let $x \in \text{supp } \mathcal{C}(\mathcal{R}, \pi) = \pi(\text{supp } \mathcal{R}) = \pi(\text{supp } \mathcal{S}) = \text{supp } \mathcal{C}(\mathcal{S}, \pi)$

$$\begin{aligned} \sigma_{\mathcal{R},\pi}(x) &:= \bigcap_{R \in \mathcal{R} \text{ s.t. } x \in \pi(R)} \pi(R) \subset \bigcap_{R \in \mathcal{R} \text{ s.t. } x \in \pi(R)} \pi(S_R) \\ &\subset \bigcap_{S \in \mathcal{S} \text{ s.t. } x \in \pi(S)} \pi(S) =: \sigma_{\mathcal{S},\pi}(x) \in \mathcal{C}(\mathcal{S}, \pi) \end{aligned}$$

□

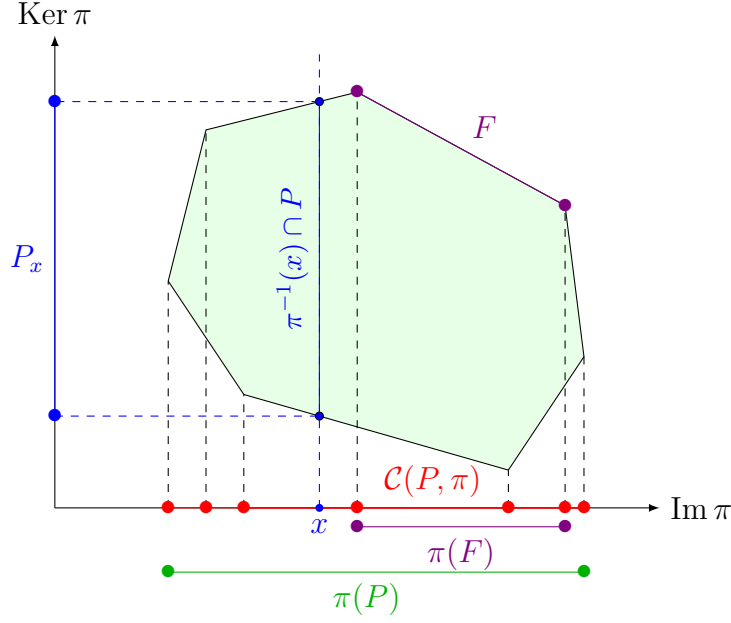


Figure 2: A polytope P in light green, its chamber complex in red on the x -axis and a fiber P_x in blue on the y -axis, for the orthogonal projection π on the horizontal axis.

Recall that the *fiber* P_x of P along π at x is the projection of $P \cap \pi^{-1}(\{x\})$ on the space $\text{Ker}(\pi)$ (see figure 2). An important property of a chamber complex is that all fibers are normally equivalent in each relative interior of cells of the chamber complex. More precisely, let $\sigma \in \mathcal{C}(P, \pi)$ be a chamber, and x and x' two points in its relative interior, then, P_x and $P_{x'}$ are *normally equivalent*, i.e. they have the same normal fan $\mathcal{N}(P_x) = \mathcal{N}(P_{x'})$, see [BS92]. Thus we define the normal fan \mathcal{N}_σ above³ $\sigma \in \mathcal{C}(P, \pi)$ by :

$$\mathcal{N}_\sigma := \mathcal{N}(P_x) \quad \text{for an arbitrary } x \in \text{ri}(\sigma)$$

The terms *parametrized polyhedron*, instead of fibers, and *validity domains*, instead of chambers, are also used in the literature [CL98, LW97].

3 Exact quantization of the 2-stage problem

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathbf{c} \in L_1(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^m)$ be an integrable random vector, and suppose $\xi = (T, W, h)$ is deterministic. We study the cost-to-go function of the 2-stage stochastic problem, written as

$$V(x) := \mathbb{E} \left[\hat{V}(x, \mathbf{c}) \right] \quad \text{with} \quad \hat{V}(x, c) := \min_{y \in \mathbb{R}^m} c^\top y \quad (4)$$

$$\text{s.t.} \quad Ay + Bx \leq b$$

³The normal fan $\mathcal{N}_\sigma \subset 2^{\mathbb{R}^m}$ above σ should not be confused with $\mathcal{N}(\sigma) \subset 2^{\mathbb{R}^n}$ the normal fan of σ which will never appear in this paper.

The dual of the latter problem, for given x and c , is

$$\begin{aligned} \max_{\mu \in \mathbb{R}^q} \quad & (Bx - b)^\top \mu \\ \text{s.t.} \quad & A^\top \mu = -c \\ & \mu \geq 0 \end{aligned} \tag{5}$$

We denote the *coupling constraint polyhedron* of Problem (4) by

$$P := \{(x, y) \in \mathbb{R}^{n+m} \mid Ay + Bx \leq b\}$$

and π the projection of $\mathbb{R}^n \times \mathbb{R}^m$ onto \mathbb{R}^n such that $\pi(x, y) = x$.

The projection of P is the following polyhedron :

$$\pi(P) = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m, Ay + Bx \leq b\} \tag{6}$$

and for any $x \in \mathbb{R}^n$, the *fiber* of P along π is

$$P_x := \{y \in \mathbb{R}^m \mid Ay + Bx \leq b\}$$

3.1 Chamber complexes arising from 2-stage problems

The following lemma provides an explicit formula for the cost-to-go function. It shows that an optimal recourse can be chosen as a function of c that is piecewise constant on the normal fan of P_x .

Lemma 8. *Let $x \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$,*

1. *If $x \notin \pi(P)$, then $\hat{V}(x, c) = +\infty$;*
2. *If $x \in \pi(P)$ and $-c \notin \text{Cone}(A^\top)$, then $\hat{V}(x, c) = -\infty$;*
3. *Suppose now that $x \in \pi(P)$ and $-c \in \text{Cone}(A^\top)$. For each cone $N \in \mathcal{N}(P_x)$, let us select in an arbitrary manner a vector c_N in $\text{ri}(-N)$. Then, there exists a vector $y_N(x)$ which achieves the minimum in the expression of $\hat{V}(x, c_N)$ in (4). Further, for any selection of such a $y_N(x)$, we have*

$$\hat{V}(x, c) = \sum_{N \in \mathcal{N}(P_x)} \mathbf{1}_{c \in -\text{ri}N} c^\top y_N(x) . \tag{7}$$

Proof. The first point comes from the definitions of $\pi(P)$ in (6) and $\hat{V}(x, c)$ in (4). If $x \in \pi(P)$ and $-c \notin \text{Cone}(A^\top)$, then the primal problem Eq. (4) is feasible and the dual problem is Eq. (5) unfeasible. Thus, by strong duality, $\hat{V}(x, c) = -\infty$.

By Eq. (3), we have that $\text{rc}(P_x)^\circ = \text{supp}(\mathcal{N}(P_x))$. Further, by Eq. (2) all non empty fibers P_x have the same recession cone $\{y \mid Ay \leq 0\}$ whose polar is $\text{Cone}(A^\top)$.

Assume now that $x \in \pi(P)$ and $-c \in \text{Cone}(A^\top) = \text{supp}(\mathcal{N}(P_x))$. Then, there exists $N \in \mathcal{N}(P_x)$ such that $-c \in \text{ri}(N)$. Moreover, for every choice of $c_N \in -\text{ri}(N)$, we have $\arg \min_{y \in P_x} c_N^\top y = \arg \min_{y \in P_x} c^\top y$, see e.g. [LR08, Cor. 1(c)]. Moreover, there exists $y_N(x)$ such that $N = N_{P_x}(y_N(x))$ by definition of a normal cone, thus $y_N(x) \in \arg \min_{y \in P_x} c_N^\top y$; in particular, the latter argmin is non empty. Thus, when $-c \in \text{ri}(N)$, $\hat{V}(x, c) = c^\top y_N(x)$.

Thanks to the partition property of Proposition 2, we know that c belongs to the relative interior of precisely one cone in the normal fan of P_x , leading to (7). \square

Having this property in mind, we make the following assumption:

Assumption 1. *The cost $\mathbf{c} \in L^1(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^m)$ is integrable with $\mathbf{c} \in -\text{Cone}(A^\top)$ almost surely.*

Theorem 9 (Quantization of the cost distribution). *Recall that $\mathcal{C}(P, \pi)$ is the chamber complex of the coupling constraint polyhedron P along the projection π on the x -space. Let $x \in \pi(P)$, and σ be a cell of $\mathcal{C}(P, \pi)$ such that $x \in \text{ri}(\sigma)$.*

Under Assumption 1, for every refinement \mathcal{R} of $-\mathcal{N}_\sigma$, we have:

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \hat{V}(x, \check{c}_R) \quad \text{with} \quad \hat{V}(x, \check{c}_R) := \min_{y \in \mathbb{R}^m} \check{c}_R^\top y + \mathbb{I}_{Ay+Bx \leq b} \quad (8)$$

where $\check{p}_R := \mathbb{P}[\mathbf{c} \in \text{ri}(R)]$ and $\check{c}_R := \mathbb{E}[\mathbf{c} \mid \mathbf{c} \in \text{ri}(R)]$ if $\check{p}_R > 0$ and $\check{c}_R := 0$ if $\check{p}_R = 0$.

In particular, if \mathcal{R} is a refinement of $\bigwedge_{\sigma \in \mathcal{C}(P, \pi)} -\mathcal{N}_\sigma$, Eq. (8) holds for all $x \in \pi(P)$.

This is an exact quantization result, since (8) shows that $V(x)$ coincides with the value function of a second stage problem with a cost distribution supported by the finite set $\{\check{c}_R \mid R \in \mathcal{R}\}$.

Proof. Let $\sigma \in \mathcal{C}(P, \pi)$ and $x \in \text{ri}(\sigma)$ then, by definition, $\mathcal{N}(P_x) = \mathcal{N}_\sigma$.

For $R \in \mathcal{R}$, there exists one and only one $N \in -\mathcal{N}_\sigma$ such that $\text{ri}(R) \subset \text{ri}(N)$, that we denote $N(R)$. Indeed, as \mathcal{R} is a refinement of $-\mathcal{N}_\sigma$, there exists at least one, and as $-\mathcal{N}_\sigma$ is a polyhedral complex it is unique.

By Lemma 8, under Assumption 1 and since $x \in \pi(P)$,

$$\begin{aligned} V(x) &= \mathbb{E} \left[\sum_{N \in \mathcal{N}(P_x)} \mathbb{1}_{\mathbf{c} \in -\text{ri} N} \mathbf{c}^\top y_N(x) \right] \\ &= \mathbb{E} \left[\sum_{N \in -\mathcal{N}_\sigma} \sum_{R \in \mathcal{R} \mid \text{ri}(R) \subset \text{ri}(N)} \mathbb{1}_{\mathbf{c} \in \text{ri} R} \mathbf{c}^\top y_N(x) \right] && \text{by the partition property} \\ &= \sum_{R \in \mathcal{R}} \mathbb{E} \left[\mathbb{1}_{\mathbf{c} \in \text{ri} R} \mathbf{c}^\top \right] y_{N(R)}(x) && \text{by linearity} \\ &= \sum_{R \in \mathcal{R}} \check{p}_R \check{c}_R^\top y_{N(R)}(x) \\ &= \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in \mathbb{R}^m} \check{c}_R^\top y + \mathbb{I}_{Ay+Bx \leq b} \end{aligned}$$

the last equality is obtained by definition of $y_{N(R)}(x)$ as $\check{c}_R \in N(R)$, which leads to Eq. (8). \square

Note that $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}^{\max}(P, \pi)} -\mathcal{N}_\sigma$ satisfies the condition of Theorem 9 since if τ is a face of σ in the chamber complex, \mathcal{N}_σ refines \mathcal{N}_τ by [RZ96, Lemma 2.2].

Corollary 10. *Under Assumption 1, let $x \in \pi(P)$ and $\sigma \in \mathcal{C}(P, \pi)$ such that $x \in \text{ri}(\sigma)$, then for every refinement \mathcal{R} of $-\mathcal{N}_\sigma$, the subgradient of V at point x is given by the Minkowski sum*

$$\partial V(x) = \sum_{R \in \mathcal{R}} \check{p}_R BD(x, \check{c}_R)$$

where $D(x, c) := \text{argmax} \{(Bx - b)^\top \mu : A^\top \mu = -c, \mu \geq 0\}$, $\check{p}_R := \mathbb{P}[\mathbf{c} \in \text{ri}(R)]$ and $\check{c}_R := \mathbb{E}[\mathbf{c} \mid \mathbf{c} \in \text{ri}(R)]$ if $\check{p}_R > 0$ and $\check{c}_R := 0$ if $\check{p}_R = 0$.

In particular, if \mathcal{R} is a refinement of $\bigwedge_{\sigma \in \mathcal{C}(P, \pi)} -\mathcal{N}_\sigma$, the subgradient formula Eq. (8) holds for all $x \in \pi(P)$.

Proof. This is a consequence of the quantization result Theorem 9 and the formula of the subgradient of the expected cost-to-go function (see e.g. [SDR14, 2.36]), taking into account the form of the dual problem Eq. (5). \square

Theorem 11 (Affine regions). *For all distributions of \mathbf{c} satisfying Assumption 1, the expected cost-to-go function V is affine on each cell of the chamber complex $\mathcal{C}(P, \pi)$.*

Proof. Let $\sigma \in \mathcal{C}(P, \pi)$. We show that for every $c \in -\text{Cone}(A^\top)$, $x \mapsto \hat{V}(x, c)$ is an affine function on $\text{ri}(\sigma)$. By [RZ96] (see Lemma 2.1 (iii) and the comment after this lemma), there exists a unique minimal face of P , denoted $F_{\sigma, c}$ that contains $\{x\} \times P_x^c$ for all $x \in \text{ri}(\sigma)$. Thus, for $x \in \text{ri}(\sigma)$, $\hat{V}(x, c) = c^\top y$ for every $y \in P_x^c$ or equivalently for every $(x, y) \in F_{\sigma, c}$. Since, $F_{\sigma, c}$ is a face, there exists an affine selection $x \mapsto y(x)$ on $\text{ri}(\sigma) \subset \pi(F_{\sigma, c})$ such that $(x, y(x)) \in F_{\sigma, c}$. Then, $\hat{V}(\cdot, c) = c^\top y(\cdot)$ is affine on $\text{ri}(\sigma)$. By the quantization result, $V(\cdot) = \sum_{R \in \mathcal{R}} \check{p}_R \hat{V}(\cdot, \check{c}_R)$ is affine on $\text{ri}(\sigma)$. \square

Remark 12. *It follows from this theorem that, for all $x \in \pi(P)$,*

$$V(x) = \max_{\sigma \in \mathcal{C}^{\max}(P, \pi)} \alpha_\sigma^\top x + \beta_\sigma \text{ with } \alpha_\sigma = \sum_{N \in -\mathcal{N}_\sigma} B^\top \mu_\sigma(\check{c}_N) \text{ and } \beta_\sigma = \sum_{N \in -\mathcal{N}_\sigma} -b^\top \mu_\sigma(\check{c}_N)$$

where $\mu_\sigma(\check{c}_N) \in D(x, \check{c}_N)$ (defined in Corollary 10) for $x \in \text{ri}(\sigma)$.

3.2 Alternative approach in terms of regular subdivisions

The exact 2-stage quantization theorem, Theorem 9, provides a polyhedral description of the expected cost-to-go function without recourse. We next present a combinatorial interpretation, through regular subdivisions and triangulations, based on a result of Walkup and Wets, describing the piecewise linear behavior of the value function of a deterministic linear program. This provides further insight, and also leads to an alternative way to prove Theorem 9. However, this section is not used in the rest of the paper.

We first recall basic notions, concerning the secondary fan, regular subdivisions and triangulations, referring to the monograph of De Loera, Rambau and Santos [DLRS10] for background.

Let us denote by $(a_i)_{i \in [q]}$ the rows of the matrix A , and let us choose $b \in \mathbb{R}^q$. We shall think of $(a_i^\top)_{i \in [q]}$ as a *vector configuration* in \mathbb{R}^m , and b as a *height vector*: for each $i \in [q]$, we draw the point $(a_i^\top, b_i) \in \mathbb{R}^m \times \mathbb{R}$. We now consider the convex hull E of the points (a_i^\top, b_i) in $\mathbb{R}^m \times \mathbb{R}$. The *geometric regular subdivision* induced by the height vector b is the polyhedral complex defined as the projection onto \mathbb{R}^m of the lower faces of the polyhedron E (i.e., the faces of E that have a normal vector with a negative ultimate coordinate). This geometric notion can be translated in terms of a notion of *combinatorial regular subdivision*, representing a face by the set of points that it contains. These notions are formalized by the following definition.

Definition 13 (Regular subdivisions, triangulations and secondary fan). *Let us denote by $(a_i)_{i \in [q]}$ the rows of the matrix A , and let $b \in \mathbb{R}^q$. The (combinatorial) regular subdivision of the configuration of vectors $(a_i^\top)_{i \in [q]}$ induced by the height vector b is the collection $\mathcal{S}(A^\top, b)$ of subsets of $[q]$ such that*

$$\mathcal{S}(A^\top, b) := \{I \subset [q] \mid \exists y \in \mathbb{R}^m, a_i y = b_i, \forall i \in I \text{ and } a_j y < b_j, \forall j \notin I\} .$$

A regular subdivision is a regular triangulation when every set $I \in \mathcal{S}(A^\top, b)$ yields an independent family of vectors $(a_i^\top)_{i \in I}$.

When A is fixed, the equivalence classes of $b \sim b' \iff \mathcal{S}(A^\top, b) = \mathcal{S}(A^\top, b')$ are relatively open cones. The collection of the closures of these cones constitutes a finite polyhedral fan, called the secondary fan, and denoted by $\Sigma\text{-fan}(A^\top)$.

The geometry of the cost-to-go function $\hat{V}(x, c)$, for a deterministic c , can be understood through the *basis decomposition theorem* of Walkup and Wets [WW69], see also the paper by Sturmfels and Thomas [ST97] for a more recent discussion. In particular, Lemma 1.4 in [ST97], applied to the dual problem Eq. (5) gives the following result.

Lemma 14 (Basis decomposition and subdivision, see [ST97]). *The set $D(x, c)$ of optimal solutions of the second stage dual problem Eq. (5), defined in Corollary 10, satisfies*

$$D(x, c) = \{\mu \in \mathbb{R}^q \mid A^\top \mu = -c, \mu \geq 0, \exists I \in \mathcal{S}(A^\top, b - Bx), \text{supp}(x) \subset I\} .$$

In particular, if x and x' belong to $\pi(P)$, the two following assertions are equivalent:

- (i) $b - Bx$ and $b - Bx'$ lie in the same relatively open cone of the secondary fan $\Sigma\text{-fan}(A^\top)$.
- (ii) For every $c \in -\text{Cone}(A^\top)$, $D(x, c) = D(x', c)$.

Futhermore, for $x \in \pi(P)$, the following assertions are equivalent

- (iii) $b - Bx$ lies in the interior of a maximal cone of the secondary fan $\Sigma\text{-fan}(A^\top)$.
- (iv) For every $c \in -\text{Cone}(A^\top)$, $D(x, c)$ is a singleton.
- (v) $\mathcal{S}(A^\top, b - Bx)$ is a triangulation.

Moreover, if we denote by a the affine function such that $a(x) := b - Bx$, we can show that

$$\mathcal{C}(P, \pi) = a^{-1}(\Sigma\text{-fan}(A^\top)) .$$

Together with this fact, the basis decomposition theorem may allow us to retrieve our previous results. Nevertheless, the proof of the equivalence between the chamber complex and the affine section of the secondary fan appearing above is technical. We believe that the proof presented in Section 3.1 enlightens better the geometric and polyhedral structure of the expected cost-to-go function.

Recall that $\bigwedge_{\sigma \in \mathcal{C}(P, \pi)} \mathcal{N}_\sigma$ appears in the exact quantization result for all x in Theorem 9. This fan equals the chamber complex $\mathcal{C}(\mathcal{N}(P), \pi_y^{x, y})$ of the normal fan $\mathcal{N}(P)$ of the coupling constraint polyhedron along the projection $\pi_y^{x, y} : (x, y) \mapsto y$. Moreover, it is also the normal fan of the *fiber polyhedron* $\Sigma(P, \pi(P))$ defined in [BS92]. This is no coincidence, as the dual formulation of the 2-stage problem can be understood thanks to a (simple) generalization of the fiber polytope of [BS92]. However, to extend this interpretation to the multi-stage setting, we need a more substantial generalization of fiber polytopes, taking into account nonanticipativity constraints and the nested structure of the control problem. We discuss such a generalization in a subsequent work. In the next section, we develop a direct approach to the multistage problem, in terms of chamber complexes.

4 Exact quantization of the multistage problem

In this section, we show that the exact quantization result established above for a general cost distribution and deterministic constraints carries over to the case of stochastic constraints with finite support and then to multistage programming.

We denote by $\pi_x^{x,y}$ for the projection from $\mathbb{R}^n \times \mathbb{R}^m$ to \mathbb{R}^n defined by $\pi_x^{x,y}(x', y') = x'$. The projections $\pi_{x,y}^{x,y,z}$, $\pi_x^{x,y,z}$, $\pi_y^{y,z}$, $\pi_{x_{t-1}}^{x_{t-1},z}$ are defined accordingly. Note that in the notation $\pi_x^{x,y,z}$, x , y and z are part of the notation and not parameters.

4.1 Propagating chamber complexes through Dynamic Programming

We next show that chamber complexes are propagated through dynamic programming in a way that is *uniform* with respect to the cost distribution. This is a key tool to extend the exact quantization theorem to the multistage setting. Note that the proof of Theorem 9 cannot be extended to the multistage setting as, in this case, the extensive form requires non-anticipativity constraints that cannot be tackled directly.

Recall that, for a polyhedron P and a vector ψ , we denote $P^\psi := \arg \min_{x \in P} \psi^\top x$. Let f be a polyhedral function on \mathbb{R}^d , with a slight abuse of notation we denote $\text{epi}(f)^{\psi,1} = \arg \min_{(x,z) \in \text{epi}(f)} \psi^\top x + z$. We denote $\mathcal{F}_{\text{low}}(\text{epi}(f)) := \{\text{epi}(f)^{\psi,1} \mid \psi \in \mathbb{R}^d\}$ the set of *lower faces* of $\text{epi}(f)$. The collection of projections (on \mathbb{R}^d) of lower faces of $\text{epi}(f)$ is the coarsest polyhedral complex such that f is affine on each of its cells (see [DLRS10, Chapter 2]). Moreover, we have

$$\pi_{\mathbb{R}^d}(\text{epi}(f)^{\psi,1}) = \arg \min_{x \in \mathbb{R}^d} \psi^\top x + f(x) \quad (10)$$

Lemma 15. *Let R be a polyhedral function on \mathbb{R}^m and $\mathcal{R} := \pi_y^{y,z}(\mathcal{F}_{\text{low}}(\text{epi}(R)))$ a coarsest polyhedral complex such that R is affine on each element of \mathcal{R} . Let $\xi = (A, B, b)$ be fixed and Assumption 1 holds. Define, for all $x \in \mathbb{R}^n$*

$$\begin{aligned} Q(x, y) &:= R(y) + \mathbb{I}_{Ay+Bx \leq b} \\ V(x) &:= \mathbb{E} \left[\min_{y \in \mathbb{R}^m} \mathbf{c}^\top y + Q(x, y) \right] \end{aligned}$$

Let $\mathcal{V} := \mathcal{C}(\mathcal{F}(P) \wedge (\mathbb{R}^n \times \mathcal{R}), \pi_x^{x,y}) \subset 2^{\mathbb{R}^n}$ with $P := \{(x, y) \mid Ay + Bx \leq b\}$.

Then, $\mathcal{V} \preceq \mathcal{C}(\text{epi}(Q), \pi_x^{x,y,z})$ and V is a polyhedral function which is affine on each element of \mathcal{V} .

Proof. We have $\text{epi}(Q) = (\mathbb{R}^n \times \text{epi}(R)) \cap (P \times \mathbb{R}) \subset \mathbb{R}^{n+m+1}$. Since

$$V(x) = \mathbb{E} \left[\min_{y \in \mathbb{R}^m, z \in \mathbb{R}} \mathbf{c}^\top y + z + \mathbb{I}_{(x,y,z) \in \text{epi}(Q)} \right],$$

by Theorem 11 applied to the problem with variables (y, z) and the coupling polyhedron $\text{epi}(Q)$, V is a polyhedral function affine on each element of $\mathcal{C}(\text{epi}(Q), \pi_x^{x,y,z})$. We now show that $\mathcal{V} \preceq \mathcal{C}(\text{epi}(Q), \pi_x^{x,y,z})$. As $\text{epi}(Q)$ is the epigraph of a polyhedral function, $\mathcal{Q} := \pi_{x,y}^{x,y,z}(\mathcal{F}_{\text{low}}(\text{epi}(Q))) \subset 2^{\mathbb{R}^{n+m}}$ is a polyhedral complex.

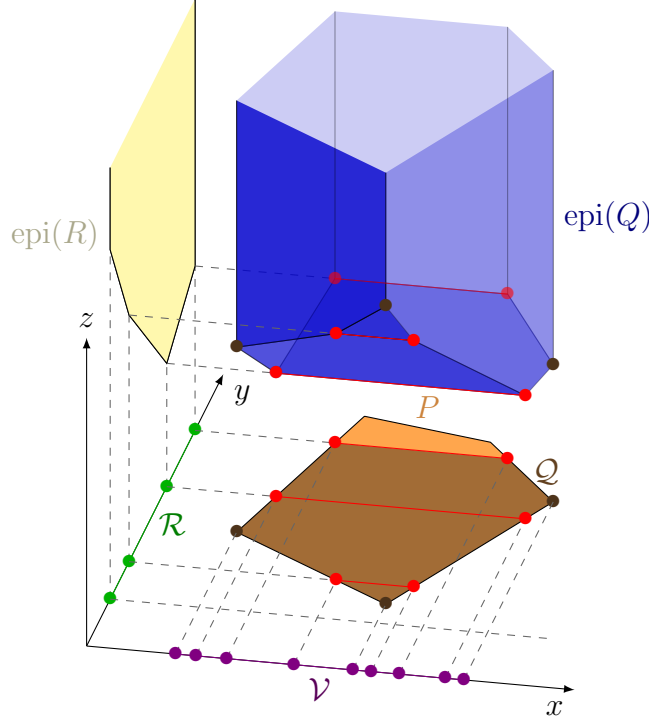


Figure 3: An illustration of the proof of Lemma 15 : the epigraph $\text{epi}(Q)$ of the coupling function in blue in the (x, y, z) space, the epigraph of R in yellow in the (y, z) plane, the affine regions \mathcal{R} of R in green on the y axis, the coupling polyhedron P in orange and brown in the (x, y) plane, the polyhedral complex \mathcal{Q} in red and brown in the (x, y) plane and the chamber complex \mathcal{V} in violet on the x axis.

Let $x_0 \in \pi_x^{x,y,z}(\text{epi}(Q))$, using notation of Definition 6,

$$\begin{aligned}
\sigma_{\text{epi}(Q), \pi_x^{x,y,z}}(x_0) &:= \bigcap_{F \in \mathcal{F}(\text{epi}(Q)) \text{ s.t. } x_0 \in \pi_x^{x,y,z}(F)} \pi_x^{x,y,z}(F) \\
&= \bigcap_{F \in \mathcal{F}_{\text{low}}(\text{epi}(Q)) \text{ s.t. } x_0 \in \pi_x^{x,y,z}(F)} \pi_x^{x,y,z}(F) \\
&= \bigcap_{F' \in \mathcal{Q} \text{ s.t. } x_0 \in \pi_x^{x,y}(F')} \pi_x^{x,y}(F') =: \sigma_{\mathcal{Q}, \pi_x^{x,y}}(x_0)
\end{aligned}$$

Indeed, as $\text{epi}(Q)$ is an epigraph of a polyhedral function, if $F \in \mathcal{F}(\text{epi}(Q))$ such that $x_0 \in \pi_x^{x,y,z}(F)$ then there exists $G \in \mathcal{F}_{\text{low}}(\text{epi}(Q))$ such that $G \triangleleft F$ and $x_0 \in \pi_x^{x,y,z}(G)$, allowing us to go from the first to second equality. The third equality is obtained by setting $F' = \pi_{x,y}^{x,y,z}(F)$. Thus, $\mathcal{C}(\text{epi}(Q), \pi_x^{x,y,z}) = \mathcal{C}(\mathcal{Q}, \pi_x^{x,y})$.

We now show that $\mathcal{F}(P) \wedge (\mathbb{R}^n \times \mathcal{R}) \preceq \mathcal{Q}$. Let $G \in \mathcal{F}(P) \wedge (\mathbb{R}^n \times \mathcal{R})$. There exist $\sigma \in \mathcal{R}$ and $F \in \mathcal{F}(P)$ such that $G = F \cap (\mathbb{R}^n \times \sigma)$. By definition of \mathcal{F}_{low} , there exists $\psi \in \mathbb{R}^m$ such that $\sigma = \pi_y^{y,z}(\text{epi}(R)^{\psi,1})$. We show that $G \subset \pi_{x,y}^{x,y,z}(\text{epi}(Q)^{0,\psi,1}) \in \mathcal{Q}$. Indeed, let $(x, y) \in G = F \cap (\mathbb{R}^n \times \pi_y^{y,z}(\text{epi}(R)^{\psi,1}))$. We have $(x, y) \in F \subset P$ such that $y \in \arg \min_{y' \in \mathbb{R}^m} \{\psi^\top y' + R(y')\}$. Which implies that $(x, y) \in \arg \min \{\psi^\top y' + R(y') \mid (x', y') \in P\}$. This also reads, by Eq. (10), as $(x, y) \in \pi_{x,y}^{x,y,z}(\text{epi}(Q)^{0,\psi,1})$. Thus, $G \subset \pi_{x,y}^{x,y,z}(\text{epi}(Q)^{0,\psi,1}) \in \mathcal{Q}$ leading to $\mathcal{F}(P) \wedge (\mathbb{R}^n \times \mathcal{R}) \preceq \mathcal{Q}$. Finally, by monotonicity, Lemma 7 ends the proof. \square

Remark 16. In Lemma 15, the complex \mathcal{V} is independent of the distribution of \mathbf{c} . However, for special choices of \mathbf{c} , V might be affine on each cell of a coarser complex than \mathcal{V} . For instance, if $R = 0$ and $\mathbf{c} \equiv 0$, we have that $V = \mathbb{I}_{\pi_x^{x,y}(P)}$, V is affine on $\pi_x^{x,y}(P)$. Nevertheless, $\mathcal{V} = \mathcal{C}(P, \pi_x^{x,y})$ is generally finer than $\mathcal{F}(\pi_x^{x,y}(P))$.

4.2 Exact quantization of MSLP

We next show that the multistage program with arbitrary cost distribution is equivalent to a multistage program with independent, finitely distributed, cost distributions. Further, for all step t , there exist affine regions, independent of the distributions of costs, where V_t is affine. Assumption 1 is naturally extended to the multistage setting as follows

Assumption 2. The sequence $(\mathbf{c}_t, \boldsymbol{\xi}_t)_{2 \leq t \leq T}$ is independent.⁴ Further, for each $t \in \{2, \dots, T\}$, $\boldsymbol{\xi}_t = (\mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t)$ is finitely supported, and $\mathbf{c}_t \in L^1(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^{n_t})$ is integrable with $\mathbf{c}_t \in -\text{Cone}(\mathbf{A}_t^\top)$ almost surely.

Note that Assumption 2 does not require independence between \mathbf{c}_t and $\boldsymbol{\xi}_t$. For $t \in [T]$, and $\xi = (A, B, b) \in \text{supp}(\boldsymbol{\xi}_t)$ we define the coupling polyhedron

$$P_t(\xi) := \{(x_{t-1}, x_t) \in \mathbb{R}^{n_{t-1}} \times \mathbb{R}^{n_t} \mid Ax_t + Bx_{t-1} \leq b\},$$

and consider, for $x_{t-1} \in \mathbb{R}^{n_{t-1}}$,

$$\tilde{V}_t(x_{t-1} | \xi) := \mathbb{E} \left[\min_{x_t \in \mathbb{R}^{n_t}} \mathbf{c}_t^\top x_t + V_{t+1}(x_t) + \mathbb{I}_{Ax_t + Bx_{t-1} \leq b} \mid \boldsymbol{\xi}_t = \xi \right]. \quad (12)$$

Then, the cost-to-go function V_t is obtained by

$$V_t(x_{t-1}) = \sum_{\xi \in \text{supp}(\boldsymbol{\xi}_t)} \mathbb{P}[\boldsymbol{\xi}_t = \xi] \tilde{V}_t(x_{t-1} | \xi) \quad (13)$$

The next two theorems extend the quantization results of Theorem 9 to the multistage settings.

Theorem 17 (Affine regions independent of the cost). Assume that $(\boldsymbol{\xi}_t)_{t \in [T]}$ is a sequence of independent, finitely supported, random variables. We define by induction $\mathcal{P}_{T+1} := \{\mathbb{R}^{n_T}\}$ and for $t \in \{2, \dots, T\}$

$$\begin{aligned} \mathcal{P}_{t,\xi} &:= \mathcal{C}(\mathbb{R}^{n_t} \times \mathcal{P}_{t+1} \wedge \mathcal{F}(P_t(\xi)), \pi_{x_{t-1}}^{x_{t-1}, x_t}) \\ \mathcal{P}_t &:= \bigwedge_{\xi_t \in \text{supp} \boldsymbol{\xi}_t} \mathcal{P}_{t,\xi} \end{aligned}$$

Then, for all costs distributions $(\mathbf{c}_t)_{2 \leq t \leq T}$ such that $(\mathbf{c}_t, \boldsymbol{\xi}_t)_{2 \leq t \leq T}$ satisfies Assumption 2 and all $t \in \{2, \dots, T\}$, we have $\text{supp}(\mathcal{P}_t) = \text{dom}(V_t)$, and V_t is polyhedral and affine on each cell of \mathcal{P}_t .

Proof. We set for all $t \in \{2, \dots, T+1\}$, $\mathcal{V}_t := \pi_{x_{t-1}}^{x_{t-1}, z}(\mathcal{F}_{\text{low}}(\text{epi}(V_t)))$ the affine regions of V_t . As $V_{T+1} \equiv 0$ is polyhedral and affine on \mathbb{R}^{n_T} , we have $\mathcal{P}_{T+1} = \mathcal{V}_{T+1}$. Assume now that for $t \in \{2, \dots, T\}$, V_{t+1} is polyhedral and \mathcal{P}_{t+1} refines \mathcal{V}_{t+1} (i.e. V_{t+1} is affine on each cell $\sigma \in \mathcal{P}_{t+1}$).

By Lemma 15, $\tilde{V}_t(\cdot | \xi)$, defined in Eq. (12), is affine on each cell of $\mathcal{C}(\mathbb{R}^{n_t} \times \mathcal{V}_{t+1} \wedge \mathcal{F}(P_t(\xi)), \pi_{x_{t-1}}^{x_{t-1}, x_t})$ which is refined by $\mathcal{P}_{t,\xi} = \mathcal{C}(\mathbb{R}^{n_t} \times \mathcal{P}_{t+1} \wedge \mathcal{F}(P_t(\xi)), \pi_{x_{t-1}}^{x_{t-1}, x_t})$ by induction hypothesis and Lemma 7. Thus, by Eq. (13), V_t is affine on each cell of \mathcal{P}_t . In particular, V_t is polyhedral and $\mathcal{P}_t := \bigwedge_{\xi_t \in \text{supp} \boldsymbol{\xi}_t} \mathcal{P}_{t,\xi}$ refines \mathcal{V}_t . Backward induction ends the proof. \square

⁴The results can be adapted to non-independent $\boldsymbol{\xi}_t$ as long as \mathbf{c}_t is independent of $(\mathbf{c}_\tau)_{\tau < t}$ conditionally on $(\boldsymbol{\xi}_\tau)_{\tau < t}$.

By Lemma 15, we have that $\mathcal{P}_{t,\xi} \preceq \mathcal{C}(\text{epi}(Q_t^\xi), \pi_{x_{t-1}}^{x_{t-1}, x_t, z})$ where $Q_t^\xi(x_{t-1}, x_t) := V_{t+1}(x_t) + \mathbb{I}_{Ax_t + Bx_{t-1} \leq b_t}$. In particular, consider $\sigma \in \mathcal{P}_{t,\xi}$, then for all $x_{t-1} \in \text{ri}(\sigma)$, all fibers $\text{epi}(Q_t^\xi)_{x_{t-1}}$ are normally equivalent. We can then define $\mathcal{N}_{t,\xi,\sigma} := \mathcal{N}(\text{epi}(Q_t^\xi)_{x_{t-1}})$ for an arbitrary $x_{t-1} \in \text{ri}(\sigma)$.

The next result shows that we can replace the MSLP problem Eq. (1) by an equivalent problem with a discrete cost distribution.

Theorem 18 (Exact quantization of the cost distribution, Multistage case). *Assume that $(\boldsymbol{\xi}_t)_{t \in [T]}$ is a sequence of independent, finitely supported, random variables. Then, for all costs distributions such that $(\mathbf{c}_t, \boldsymbol{\xi}_t)_{2 \leq t \leq T}$ satisfies Assumption 2, for all $t \in [T]$, all $x_{t-1} \in \mathbb{R}^{n_{t-1}}$ and all $\xi \in \text{supp}(\boldsymbol{\xi}_t)$, we have a quantized version of Eq. (12):*

$$\tilde{V}_t(x_{t-1} | \xi) = \sum_{N \in \mathcal{N}_{t,\xi}} \check{p}_{t,N|\xi} \min_{x_t \in \mathbb{R}^{n_t}} \left\{ \check{c}_{t,N|\xi}^\top x_t + V_{t+1}(x_t) + \mathbb{I}_{Ax_t + Bx_{t-1} \leq b} \right\}$$

where $\mathcal{N}_{t,\xi} := \bigwedge_{\sigma \in \mathcal{P}_{t,\xi}} -\mathcal{N}_{t,\xi,\sigma}$ and for all $\xi \in \text{supp}(\boldsymbol{\xi}_t)$ and $N \in \mathcal{N}_{t,\xi}$ we denote

$$\begin{aligned} \check{p}_{t,N|\xi} &:= \mathbb{P}[\mathbf{c}_t \in \text{ri } N \mid \boldsymbol{\xi}_t = \xi] \\ \check{c}_{t,N|\xi} &:= \begin{cases} \mathbb{E}[\mathbf{c}_t \mid \mathbf{c}_t \in \text{ri } N, \boldsymbol{\xi}_t = \xi] & \text{if } \mathbb{P}[\boldsymbol{\xi}_t = \xi, \mathbf{x} \in \text{ri } N] \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Proof. Since $\tilde{V}_t(x_{t-1} | \xi) = \mathbb{E}[\min_{x_t \in \mathbb{R}^{n_t}, z \in \mathbb{R}} \mathbf{c}_t^\top x_t + z + \mathbb{I}_{(x_{t-1}, x_t, z) \in \text{epi}(Q_t^\xi)}]$ and $\mathcal{P}_{t,\xi}$ refines $\mathcal{C}(\text{epi}(Q_t^\xi), \pi_{x_{t-1}}^{x_{t-1}, x_t, z})$, by applying Theorem 9 with variables (x_t, z) and the coupling constraints polyhedron $\text{epi}(Q_t^\xi)$, we deduce that the coefficients $(\check{p}_{t,N|\xi})_{N \in \mathcal{N}_{t,\xi}}$ and $(\check{c}_{t,N|\xi})_{N \in \mathcal{N}_{t,\xi}}$ satisfy

$$\tilde{V}_t(x_{t-1} | \xi) = \sum_{N \in \mathcal{N}_{t,\xi}} \check{p}_{t,N|\xi} \min_{x_t \in \mathbb{R}^{n_t}, z \in \mathbb{R}} \left\{ \check{c}_{t,N|\xi}^\top x_t + z + \mathbb{I}_{(x_{t-1}, x_t, z) \in \text{epi}(Q_t^\xi)} \right\}$$

as the deterministic coefficient before z is equal to its conditional expectation. \square

In particular, the MSLP problem is equivalent to a finitely supported MSLP as shown in the following result.

For $t_0 \in [T - 1]$, we construct the scenario tree \mathcal{T}_{t_0} as follows. A node of depth $t - t_0$ of \mathcal{T}_{t_0} is labelled by a sequence $(N_\tau, \xi_\tau)_{t_0 < \tau \leq t}$ where $N_\tau \in \mathcal{N}_{\tau, \xi_\tau}$ and $\xi_\tau \in \text{supp}(\boldsymbol{\xi}_\tau)$. In this way, a node of depth $t - t_0$ of \mathcal{T}_{t_0} keeps track of the sequence of realizations of the random variables $\boldsymbol{\xi}_\tau$ for times τ between t_0 and t , and of a selection of cones in \mathcal{N}_{t,ξ_t} at the same times. Note that, by the independence assumption, all the subtrees of \mathcal{T}_{t_0} , starting from a node of depth $t - t_0$ are the same as \mathcal{T}_{t_0+t} . We denote by $\text{lv}(\mathcal{T}_{t_0})$ the set of leaves of \mathcal{T}_{t_0} .

Corollary 19 (Equivalent finite tree problem). *Define the quantized probability cost $c_\nu := \check{c}_{t,N_t|\xi_t}$ and probability $p_\nu := \prod_{t_0 < \tau \leq t} p_{\xi_\tau} \check{p}_{\tau, N_\tau | \xi_\tau}$, for all nodes $\nu = (N_\tau, \xi_\tau)_{t_0 < \tau \leq t}$. Then, the cost-to-go functions associated with Eq. (MSLP) are given by*

$$\begin{aligned} V_{t_0}(x_0) &= \min_{(x_\nu)_{\nu \in \mathcal{T}_{t_0}}} \sum_{\nu \in \mathcal{T}_{t_0}} p_\nu c_\nu^\top x_\nu \\ \text{s.t.} \quad & Ax_\mu + Bx_\nu \leq b \quad \forall \nu \in \mathcal{T}_{t_0} \setminus \text{lv}(\mathcal{T}_{t_0}), \forall \mu \succcurlyeq \nu, \end{aligned}$$

for all $2 \leq t_0 \leq T - 1$. Here, x_0 is the value of x at the root node of \mathcal{T}_{t_0} , and the notation $\forall \mu = (\nu, N, A, B, b) \succcurlyeq \nu$ indicates that μ ranges over the set of children of ν .

5 Computing the quantized costs and probabilities

In this section, we show that, for three standard classes of distributions (uniform on a polytope, exponential, and Gaussian), the quantized costs \check{c}_R and probabilities \check{p}_R arising in the representation of the expected cost-to-go function (Theorem 9), can be effectively computed.

The formulas are summed up in Table 1. They are detailed and established in Sections 5.1–5.3. We provide these formulas for *simplices* or *simplicial cones* S with $\dim(S) = \dim(\text{supp } \mathbf{c})$. This extends to any polyhedron R , through triangulation of $R \cap \text{supp}(\mathbf{c})$ into simplices and simplicial cones $(S_k)_{k \in [l]}$. We then compute $\check{p}_R = \sum_{k=1}^l \check{p}_{S_k}$ and $\check{c}_R = \sum_{k=1}^l \check{p}_{S_k} \check{c}_{S_k} / \check{p}_R$ if $\check{p}_R \neq 0$ and $\check{c}_R = 0$ otherwise.

	Uniform	Exponential	Gaussian
$d\mathbb{P}(c)$	$\frac{\mathbf{1}_{c \in Q}}{\text{Vol}_d(Q)} d\mathcal{L}_{\text{Aff}(Q)}(c)$	$\frac{e^{\theta^\top c} \mathbf{1}_{c \in K}}{\Phi_K(\theta)} d\mathcal{L}_{\text{Aff}(K)}(c)$	$\frac{e^{-\frac{1}{2}c^\top M^{-2}c}}{(2\pi)^{\frac{m}{2}} \det M} dc$
$\text{supp } \mathbf{c}$	Polytope : Q	Cone : K	\mathbb{R}^m
\check{p}_S	$\frac{\text{Vol}_d(S)}{\text{Vol}_d(Q)}$	$\frac{ \det(\text{Ray}(S)) }{\Phi_K(\theta)} \prod_{r \in \text{Ray}(S)} \frac{1}{-r^\top \theta}$	$\text{Ang}(M^{-1}S)$
\check{c}_S	$\frac{1}{d} \sum_{v \in \text{Vert}(S)} v$	$\left(\sum_{r \in \text{Ray}(S)} \frac{-r_i}{r^\top \theta} \right)_{i \in [m]}$	$\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} M \text{SpCtr}(S \cap \mathbb{S}_{m-1})$

Table 1: Probabilities \check{p}_S and expectations \check{c}_S arising from different cost distributions over simplicial cones or simplices $S \subset \text{supp}(\mathbf{c})$ with $\dim S = \dim(\text{supp } \mathbf{c})$, where \mathcal{L}_A is the Lebesgue measure on an affine space A .

5.1 Uniform distributions on polytopes

The *volume* of a polytope $Q \subset \mathbb{R}^m$ is the volume of P seen as a subset of the smallest affine space $\text{Aff}(Q)$ it lives in. The volume of a full dimensional simplex S in \mathbb{R}^d with vertices v_1, \dots, v_{d+1} is given by $\text{Vol}(S) = \frac{1}{n!} |\det(v_1 - v_{d+1}, \dots, v_d - v_{d+1})|$, see for example [GK94] 3.1. The *centroid* of a non-empty polytope $Q \subset \mathbb{R}^m$ is $\text{Ctr}(Q) := \frac{1}{\text{Vol}Q} \int_Q y d\mathcal{L}_{\text{Aff}Q}(y)$. For instance, the centroid of a simplex S of (non necessary full) dimension d is the equibarycenter of its vertices : $\text{Ctr}(S) = \frac{1}{d+1} \sum_{v \in \text{Vert}(S)} v$.

Assume now that Q is a polytope of dimension d , and that \mathbf{c} is uniform on Q . Let $S \subset Q$ be a simplex with $\dim(S) = \dim(Q)$, then we have

$$\check{p}_S = \frac{\text{Vol}_d S}{\text{Vol}_d Q} \quad \text{and} \quad \check{c}_S = \frac{1}{d+1} \sum_{v \in \text{Vert}(S)} v .$$

5.2 Exponential distributions on cones

Let P be a polyhedron and $\theta \in \text{ri}((\text{rc } P)^\circ)$. We denote by $\Phi_P(\theta) := \int_P e^{\theta^\top c} d\mathcal{L}_{\text{Aff}(P)}(c)$ the *exponential valuation* of P with parameter θ .

Proposition 20 (Brion's formula [Bri88]). *Let S be a full dimensional simplicial cone, and let $\text{Ray}(S)$ be a square matrix whose columns are obtained by selecting precisely one element in every extreme ray of S , so that $S = \text{Cone}(\text{Ray}(S))$. Then for any $\theta \in \text{ri } S^\circ$, the exponential valuation of S is given by*

$$\Phi_S(\theta) = |\det(\text{Ray}(S))| \prod_{r \in \text{Ray}(S)} \frac{1}{-\theta^\top r} . \quad (17)$$

Let K be a (non necessarily simplicial) polyhedral cone and $\theta \in \text{ri } K^\circ$ a vector. Assume that \mathbf{c} has the following exponential density :

$$d\mathbb{P}(\mathbf{c}) := e^{\theta^\top \mathbf{c}} \mathbf{1}_{\mathbf{c} \in K} \frac{1}{\Phi_K(\theta)} d\mathcal{L}_{\text{Aff}(K)}(\mathbf{c})$$

Let $S \subset K$ be a simplicial cone with $\dim S = \dim K$, by Brion's formula (17),

$$\check{p}_S = \frac{\Phi_S(\theta)}{\Phi_K(\theta)} = \frac{1}{\Phi_K(\theta)} |\det(\text{Ray}(S))| \prod_{r \in \text{Ray}(S)} \frac{1}{-r^\top \theta}$$

Further,

$$\check{p}_S \check{c}_S = \mathbb{E}[\mathbf{1}_{\mathbf{c} \in S} \mathbf{c}^\top] = \frac{1}{\Phi_K(\theta)} \int_S \mathbf{c} e^{\theta^\top \mathbf{c}} d\mathbf{c} = \frac{\nabla \Phi_S(\theta)}{\Phi_K(\theta)} .$$

By computing explicitly the latter gradient, dividing by \check{p}_S , and simplifying, we obtain:

$$\check{c}_S = \left(\sum_{r \in \text{Ray}(S)} \frac{-r_i}{r^\top \theta} \right)_{i \in [m]} .$$

5.3 Gaussian distributions

The *solid angle* of a pointed cone $K \subset \mathbb{R}^d$ is defined as the normalized volume of its intersection with the unit ball \mathbb{B}_d , i.e.: $\text{Ang}(K) := \text{Vol}_d(K \cap \mathbb{B}_d) / \text{Vol}_d \mathbb{B}_d$. Recall that $\text{Vol}_d \mathbb{B}_d = \pi^{\frac{d}{2}} / \Gamma(\frac{d}{2} + 1)$ with Γ the Euler gamma function, and that ([Rib06]) for any function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ invariant under rotations around the origin and any pointed cone $K \subset \mathbb{R}^m$, we have $\text{Ang}(K) \int_{\mathbb{R}^m} f = \int_K f$.

Let \mathbf{c} be a non-degenerate, centered, Gaussian random variable of variance M^2 , where M is a symmetric positive definite matrix. Then, if K is a polyhedral cone, we have

$$\check{p}_K = \int_{M^{-1}K} \frac{e^{-\frac{1}{2} \|\mathbf{c}\|_2^2}}{(2\pi)^{\frac{m}{2}}} d\mathbf{c} = \text{Ang}(M^{-1}K)$$

We shall use the notion of *spherical centroid* $\text{SpCtr}(U)$ for a measurable subset U included in the unit sphere. It is defined as the barycenter of the elements of U with respect to the uniform measure on the sphere. Note that the spherical centroid does not belong to the sphere, unless U is trivial. We have

$$\begin{aligned} \check{p}_K \check{c}_K &= \int_{M^{-1}K} M \mathbf{c} \frac{e^{-\frac{1}{2} \|\mathbf{c}\|_2^2}}{(2\pi)^{\frac{m}{2}}} d\mathbf{c} = M \int_{\mathbb{R}^+} r^m \frac{e^{-\frac{r^2}{2}}}{(2\pi)^{\frac{m}{2}}} dr \int_{M^{-1}K \cap \mathbb{S}_{m-1}} \varphi d\varphi \\ &= M \frac{\Gamma(\frac{m+1}{2})}{\sqrt{2\pi^{\frac{m}{2}}}} \text{Vol}_{m-1}(\mathbb{S}_{m-1}) \text{Ang}(M^{-1}K) \text{SpCtr}(M^{-1}K \cap \mathbb{S}_{m-1}) \\ &= M \frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} \text{Ang}(M^{-1}K) \text{SpCtr}(M^{-1}K \cap \mathbb{S}_{m-1}) \end{aligned}$$

Similarly, one can get explicit formulæ when \mathbf{c} is distributed uniformly on an ellipsoid, or on the surface of an ellipsoid, or more generally, when the distribution of \mathbf{c} is invariant under the action of an orthogonal group. Then, the quantized costs and probabilities \check{c}_S and \check{p}_S are still given by solid angles and spherical centroids, in a way similar to Table 1.

5.4 An illustrative example

We consider the following second-stage problem, with $n = 1$ and $m = 2$:

$$V(x) = \mathbb{E} \left[\begin{array}{l} \min_{y \in \mathbb{R}^2} \mathbf{c}^\top y \\ \text{s.t.} \quad \|y\|_1 \leq 1, \quad y_1 \leq x \text{ and } y_2 \leq x \end{array} \right].$$

We apply our results, to provide an explicit representation of V .

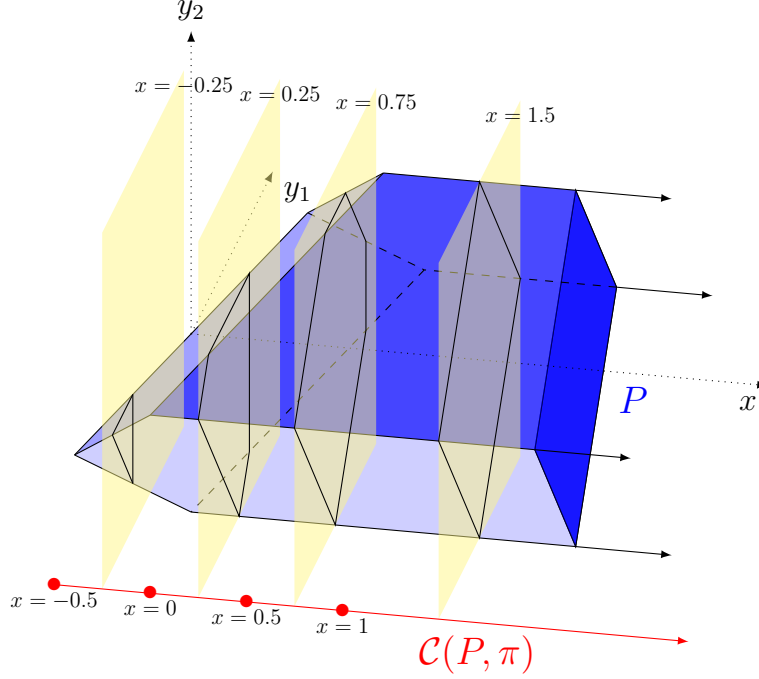
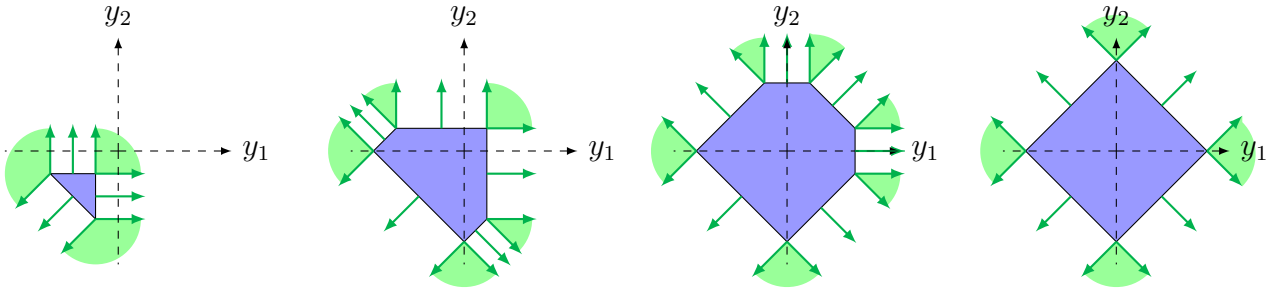


Figure 4: The coupling polyhedron P in blue, different cuts and fibers P_x vertical in yellow, and its chamber complex $\mathcal{C}(P, \pi)$ in red on the bottom.



(a) $x = -0.25, \sigma = [-0.5, 0]$ (b) $x = 0.25, \sigma = [0, 0.5]$ (c) $x = 0.75, \sigma = [0.5, 1]$ (d) $x \geq 1, \sigma = [1, +\infty)$

Figure 5: Fibers P_x in blue and their normal fan $\mathcal{N}(P_x) = \mathcal{N}_\sigma$ in green for different $x \in \mathbb{R}$

The coupling polyhedron is $P = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid \|y\|_1 \leq 1, y_i \leq x \quad \forall i \in [m]\}$ presented in Fig. 4, and its V-representation is the collection of vertices $(0, -1, 0)$, $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $(0, 0, -1)$, $(1, 1, 0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(1, 0, 1)$ and the ray $(1, 0, 0)$. By projecting the different faces, we see that its projection is the half-line, $\pi(P) = [-\frac{1}{2}, +\infty[$ and its chamber complex is $\mathcal{C}(P, \pi)$ is the collection

of cells composed of $\{-\frac{1}{2}\}, [-\frac{1}{2}, 0], \{0\}, [0, \frac{1}{2}], \{\frac{1}{2}\}, [\frac{1}{2}, 1], \{1\}, [1, +\infty)$ as presented in Fig. 4. As there are 4 different maximal chambers, there are 4 different classes of normally equivalent fibers as shown in Fig. 5.

We evaluate \check{c}_N and \check{p}_N for $N \in -\mathcal{N}_\sigma$ using the formulas of Table 1. For example, when \mathbf{c} is uniform on the centered ball for the ∞ -norm of radius R , Fig. 6 shows the regions of which the areas and centroids need to be computed.

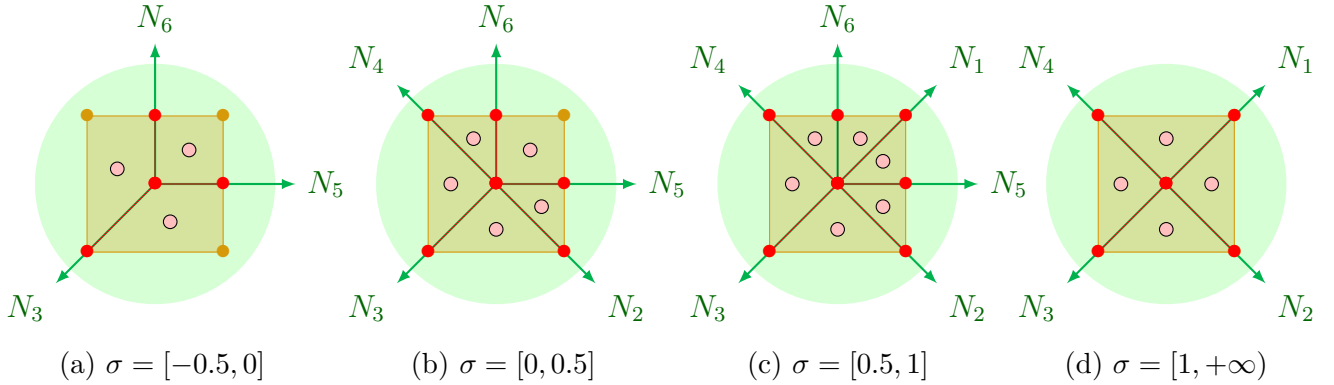


Figure 6: Exact quantization illustrated. The normal fan \mathcal{N}_σ in green with $N_i = W_i^\top \mathbb{R}^+$, \mathbf{c} is uniform on the support $Q = -Q = B_\infty(0, R)$ in light orange, the sets $W_i^\top \mathbb{R}^+ \cap Q$ in red. The polyhedral complex \mathcal{R}_σ shown in red or orange. The quantized costs \check{c}_N are determined by centroids (small circles in pink).

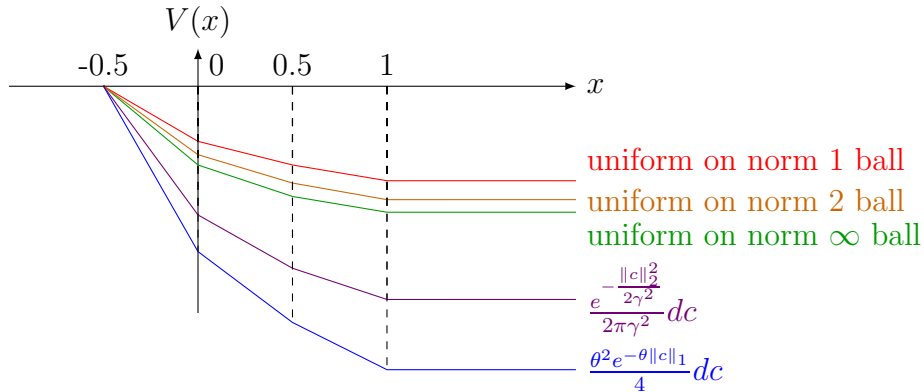


Figure 7: Graph of the cost-to-go function V for different distribution of the cost \mathbf{c} with $R = \theta = \gamma = 1$.

6 Complexity

Hanasusanto, Kuhn and Wiesemann showed in [HKW16] that 2-stage stochastic programming is $\sharp\text{P}$ -hard, by reducing the computation of the volume of a polytope to the resolution of a 2-stage stochastic program. Nevertheless, we show that for a fixed dimension of the recourse space, 2-stage programming is polynomial. Therefore, the status of 2-stage programming seems somehow comparable to the one of the computation of the volume of a polytope – which is also both $\sharp\text{P}$ -hard

$d\mathbb{P}(c)$	$-\frac{1}{2} \leq x \leq 0$	$0 \leq x \leq \frac{1}{2}$	$\frac{1}{2} \leq x \leq 1$	$1 \leq x$
$\frac{\mathbb{1}_{\ c\ _1 \leq R}}{2R^2} dc$	$\frac{-7R}{24}(1+2x)$	$\frac{-R}{24}(7+6x)$	$\frac{-R}{6}(2+x)$	$\frac{-R}{2}$
$\frac{\theta^2 e^{-\theta\ c\ _1}}{4} dc$	$\frac{-7}{8\theta}(1+2x)$	$\frac{-1}{8\theta}(7+6x)$	$\frac{-1}{2\theta}(2+x)$	$\frac{-3}{2\theta}$
$\frac{\mathbb{1}_{\ c\ _\infty \leq R}}{4R^2} dc$	$\frac{-R}{12}(5+10x)$	$\frac{-R}{12}(5+4x)$	$\frac{-R}{6}(3+x)$	$\frac{-2R}{3}$
$\frac{e^{-\ c\ _2^2/2\gamma^2}}{2\pi\gamma^2} dc$	$\frac{-\gamma(2+\sqrt{2})(1+2x)}{2\sqrt{2}\pi}$	$\frac{-\gamma(2+\sqrt{2}+2\sqrt{2}x)}{2\sqrt{2}\pi}$	$\frac{-2\gamma(1+(-1+\sqrt{2})x)}{\sqrt{2}\pi}$	$-\frac{2}{\sqrt{\pi}}\gamma$
$\frac{\mathbb{1}_{\ c\ _2 \leq R}}{\pi R^2} dc$	$\frac{-R(2+\sqrt{2})(1+2x)}{3\pi}$	$\frac{-R(2+\sqrt{2}+2\sqrt{2}x)}{3\pi}$	$\frac{-4R(1+(-1+\sqrt{2})x)}{3\pi}$	$-\frac{4\sqrt{2}R}{3\pi}$

Table 2: Different values of $V(x)$ for different distribution of the cost \mathbf{c}

and polynomial when the dimension is fixed (see for example [GK94, 3.1.1]). We also give a similar result for multistage stochastic linear programming.

We now give a summary of our method. A naive approach would be to use directly the exact quantization result Theorem 9, for every x . However, even in the two stage case, the latter yields a linear program of an exponential size when only the recourse dimension m is fixed. Indeed, the size of the quantized linear program, ($2SLP$) is polynomial only when *both* n and m are fixed. Indeed, $\bigwedge_{\sigma \in \mathcal{C}(P, \pi)} -\mathcal{N}_\sigma$ can have, by McMullen’s and Stanley’s upper bound theorems ([McM70, Sta75]), an exponential size in n and m , and these bounds are tight. Hence, to handle the case in which only the recourse dimension m is fixed, we need additional ideas. We use the quantization result Theorem 9 only for a *fixed* x , observing that when m is fixed, $\mathcal{N}(P_x)$ has a polynomial size. We thus have a polynomial time oracle that gives the values $V(x)$ by Theorem 9 and a subgradient $g \in \partial V(x)$ by Corollary 10. Then, we rely on the theory of linear programming with oracle [GLS12], working in the *Turing model* of computation (a.k.a. *bit model*). In particular, all the computations are carried out with rational numbers. We now provide the needed details of the proof.

6.1 Multistage programming with exact oracles

Recall that a polyhedron can be given in two manners. The “ H -representation” provides an external description of the polyhedron, as the intersection of finitely many half-spaces. The “ V -representation” provides an internal representation, writing the polyhedron as a Minkowski sum of a polytope (given as the convex hull of finitely many points) and of a polyhedral cone (generated by finitely many vectors).

We say that a polyhedron is *rational* if the inequalities in its H -representation are rational or, equivalently, the generators of its V -representation have rational coefficients. We shall say that a (convex) polyhedral function V is *rational* if its epigraph is a rational polyhedron.

Recall that, in the Turing model, the *size* (or encoding length see [GLS12, 1.3]) of an integer $k \in \mathbb{Z}$ is $\langle k \rangle := 1 + \lceil \log_2(|k| + 1) \rceil$; the size of a rational $r = \frac{p}{q} \in \mathbb{Q}$ with p and q coprime integers, is $\langle r \rangle := \langle p \rangle + \langle q \rangle$. The size of a rational matrix or a vector, still denoted by $\langle \cdot \rangle$, is the sum of the sizes of its entries. The size of an inequality $\alpha^\top x \leq \beta$ is $\langle \alpha \rangle + \langle \beta \rangle$. The size of a H -representation of a polyhedron is the sum of the sizes of its inequalities and the size of a V -representation of a polyhedron is the sum of the sizes of its generators.

If the dimension of the ambient space is *fixed*, one can pass from one representation to the other one in *polynomial time*. Indeed, the double description algorithm allows one to get a V -representation from a H -representation, see the discussion at the end of section 3.1 in [FP95], and use McMullen’s upper bound theorem ([McM70] and [GLS12, 6.2.4]) to show that the computation time is polynomially bounded in the size of the H -representation. A fortiori, the size of the V -

representation is polynomially bounded in the size of the H -representation. Dually, the same method allows one to obtain a H -representation from a V -representation. Hence, in the sequel, we shall use the term *size* of a polyhedron for the size of a V or H -representation: when dealing with polynomial-time complexity results in fixed dimension, whichever representation is used is irrelevant. In particular, we define the *size* $\langle N \rangle$ of a rational cone N as the size of a H or V representation of N .

We first observe that the size of the scenario tree arising in the exact quantization result becomes polynomial when suitable dimensions are fixed.

Proposition 21. *Let $t \in \{2, \dots, T\}$, and suppose that the dimensions n_2, \dots, n_T and the cardinals $\sharp(\text{supp } \xi_2), \dots, \sharp(\text{supp } \xi_T)$ are fixed. Let \mathcal{T} be the scenario tree constructed in Corollary 19. Then, the subtree of \mathcal{T} rooted at an arbitrary node of depth t can be computed in polynomial in $\sum_{s=t}^T \sum_{\xi \in \text{supp}(\xi_s)} \langle \xi \rangle$.*

Proof. Recall that the number of chambers of a chamber complex is polynomial when both dimensions are fixed by [VWBC05, 3.9]. Thus, we can compute recursively the (maximal) chambers of the complexes \mathcal{P}_t defined in Theorem 17 thanks to the algorithm in [CL98, 3.2] in polynomial time. We then can compute in polynomial time the fans \mathcal{N}_t defined in Theorem 18. \square

We recall the theory of linear programming with oracle applies to the class of “well described” polyhedra which are rational polyhedra with an a priori bound on the bit-sizes of the inequalities defining their facets, we refer the reader to [GLS12] for a more detailed discussion of the notions (oracles) and results used here.

Definition 22 (first-order oracle). *Let f be a rational polyhedral function. We say that f admits a polynomial time (exact) first-order oracle, if there exists an oracle that takes as input a vector x and either returns a hyperplane separating x from $\text{dom}(f)$ if $x \notin \text{dom}(f)$ or returns $f(x)$ and $g \in \partial V(x)$ if $x \in \text{dom}(f)$, in polynomial time in $\langle x \rangle$.*

Lemma 23. *Let $Q \subset \mathbb{R}^d$ be a polyhedron, $c \in \mathbb{R}^d$ a cost vector and f be a polyhedral function given by a first-order oracle. Furthermore, assume $\text{epi}(f)$ and Q are well described. Then, the problem $\min_{x \in Q} c^\top x + f(x)$ can be solved in oracle-polynomial time in $\langle c \rangle + \langle \text{epi}(f) \rangle + \langle Q \rangle$.*

Proof. The case where $\text{dom}(f) = \mathbb{R}^d$ is tackled in Theorem 6.5.19 in [GLS12]. If f has a general domain, we can write $f = \tilde{f} + \mathbb{I}_{\text{dom } f}$ where \tilde{f} is a polyhedral function with a well described epigraph and such that $\text{dom } \tilde{f} = \mathbb{R}^d$. Then, noting that $\text{epi}(f) = \text{epi}(\tilde{f}) \cap \text{dom}(f) \times \mathbb{R}$, we can adapt the proof of the latter theorem, using Exercise 6.5.18(a) of [GLS12]. \square

We do not require the distribution of the cost c to be described extensively. We only need to assume the existence of the following oracle.

Definition 24 (cone-valuation oracle). *Let $c \in L(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{R}^m)$ be an integrable cost distribution such that, for every rational cone N , the quantized probability \check{p}_N and quantized cost \check{c}_N are rational. We say that c admits a polynomial time (exact) cone-valuation oracle, if there exists an oracle which takes as input a rational polyhedral cone N and returns \check{p}_N and \check{c}_N in polynomial time in $\langle N \rangle$.*

Theorem 25 (Cone valuation to first-order oracle). *Consider the value functions of problem (MSLP) defined in Eq. (1). Assume that $T, n_2, \dots, n_T, \sharp(\text{supp } \xi_2), \dots, \sharp(\text{supp } \xi_T)$ are fixed integers, and that $(c_t, \xi_t)_{2 \leq t \leq T}$ satisfies Assumption 2. Assume in addition that, every vector*

$\xi \in \text{supp}(\xi_t)$ has rational entries and that the probabilities $p_{t,\xi} := \mathbb{P}[\xi_t = \xi]$ are rational numbers. Assume finally that every random variable \mathbf{c}_t conditionally to $\{\xi_t = \xi\}$, denoted by $\mathbf{c}_{t,\xi}$, admits a polynomial-time cone-valuation oracle (see Definition 24).

Then, for all $t \geq 2$, V_t admits a polynomial time first-order oracle.

Proof. We start with the 2-stage case with deterministic constraints. We recall our notation $V(x) := \mathbb{E}[\min_{y \in \mathbb{R}^m} \mathbf{c}^\top y + \mathbb{I}_{Ay+Bx \leq b}]$. Let $x \in \mathbb{R}^n$ be an input vector. We first check if $x \in \pi(P) = \text{dom}(V)$. By solving the dual of $\min_{y \in \mathbb{R}^q} \{0 \mid Ay \leq b - Bx\}$, we either find an unbounded ray generated by $\lambda \in \mathbb{R}^q$ such that $\lambda \geq 0$, $\lambda^\top A = 0$ and $\lambda^\top (b - Bx) < 0$ or a $y \in \mathbb{R}^m$ such that $Ay \leq b - Bx$, so that $x \in \pi(P)$. In the former case we have $x \notin \pi(P)$, and we get a cut $\{x' \in \mathbb{R}^n \mid \lambda^\top Bx' = \frac{\lambda^\top b + \lambda^\top Ax}{2}\}$, separating $\pi(P) = \text{dom}(V)$ from x .

So, we now assume that $x \in \pi(P)$, i.e., $V(x) < +\infty$. We next show that we can compute $V(x)$ and a subgradient $\alpha \in \partial V(x)$ in polynomial time. Indeed, the McMullen upper-bound theorem [McM70], in its dual version, guarantees that a polytope of dimension m with f facets has $O(f^{\lfloor m/2 \rfloor})$ faces, see [Sei95]. Since the number of cones in $\mathcal{N}(P_x)$ is equal to the number of faces of P_x which is polynomially bounded in the number of constraints $q \leq \langle \xi \rangle$, $\#\mathcal{N}(P_x)$ is polynomial in $\langle \xi \rangle$. Thus, since \mathbf{c} is given by a cone valuation oracle, we can compute in polynomial time the collection of all quantized costs and probabilities \check{c}_N and \check{p}_N , indexed by $N \in -\mathcal{N}(P_x)$. Then, by Theorem 9, we can compute $V(x)$ by solving a linear program for each cone $N \in -\mathcal{N}(P_x)$. Similarly, Corollary 10, allows us to compute a subgradient $\alpha \in \partial V(x)$ using the same linear programs. All these operations take a polynomial time.

The case of finitely supported stochastic constraints reduces to the case of deterministic constraints dealt with above, using $\text{dom}(V) = \bigcap_{\xi \in \text{supp } \xi} \pi(P(\xi))$ and $V(x) = \sum_{\xi \in \text{supp } \xi} p_\xi \tilde{V}(x|\xi)$ where $\tilde{V}(x|\xi) := \mathbb{E}[\hat{V}(x, \mathbf{c}, \xi) \mid \xi = \xi]$.

We finally deal with the multistage case in a similar way, using the quantization result Corollary 19 in extensive form. Applying Proposition 21, the quantized costs and probabilities arising there can be computed by a polynomial number of calls to the cone-valuation oracle. This provides a first order oracle for the expected cost-to-go function V_t . □

We now refine the definition of cone-valuation oracle, to take into account situations in which the distribution of the random cost \mathbf{c} is specified by a parametric model. We shall say that such a distribution admits a polynomial-time *parametric cone-valuation oracle* if there is an oracle that takes as input the parameters of the distribution, together with a rational cone N , and outputs the quantized probability \check{p}_N and cost \check{c}_N . Especially, we consider the following situations:

1. *Deterministic distribution* equal to a rational cost c . We set $\langle \mathbf{c} \rangle := \langle c \rangle$
2. *Exponential distribution on a rational cone* K with rational parameter θ . We set $\langle \mathbf{c} \rangle := \langle K \rangle + \langle \theta \rangle$
3. *Uniform distribution on a rational polyhedron* Q such that $\text{Aff}(Q) = \{y \in \mathbb{R}^m \mid \forall j \in J \subset [m], y_j = q_j \in \mathbb{Q}\}$ where J is a subset of $[m]$ and q_j are rational numbers (in particular, Q is full dimensional when $J = \emptyset$). We set: $\langle \mathbf{c} \rangle = \langle Q \rangle$
4. *Mixtures of the above distributions*, i.e., convex combination with rational coefficients $(\lambda^k)_{k \in [l]}$ of distributions of random variables $(\mathbf{c}_k)_{k \in [l]}$ satisfying 1. 2. or 3. Then, we set $\langle \mathbf{c} \rangle = \sum_{k=1}^l \langle \mathbf{c}_k \rangle + \langle \lambda_k \rangle$.

Theorem 26. *Assume that the dimension m is fixed, and that \mathbf{c} is distributed according to any of the above laws (deterministic, exponential, uniform, or mixture). Then, the random cost \mathbf{c} admits a polynomial-time parametric cone-valuation oracle.*

Proof. 1. *Case of a deterministic distribution.* We first check whether $c \in \text{ri}(N)$, which can be done in polynomial time, see section 6.5 of [GLS12]. Then, if $c \in \text{ri}(N)$, we set $\check{c}_N = c$ and $\check{p}_N = 1$ otherwise $\check{c}_N = 0$ and $\check{p}_N = 0$.

2. *Case of an exponential distribution.* Since the dimension is fixed, for every polyhedron R , we can triangulate $R \cap \text{supp}(\mathbf{c})$ and partition it into (relatively open) simplices and simplicial cones $(S_k)_{k \in [l]}$, and by Stanley upper bound theorem, the size l of the triangulation is polynomial in $\langle R \rangle$. By using the Brion formula in Table 1, we compute in polynomial time $\check{p}_R = \sum_{k=1}^l \check{p}_{S_k}$ and $\check{c}_R = \sum_{k=1}^l \check{p}_{S_k} \check{c}_{S_k} / \check{p}_R$ if $\check{p}_R = 0$ and $\check{c}_R = 0$ otherwise.

3. *Case of a uniform distribution.* After triangulating (as in the case of an exponential distribution), we may suppose that the support of the distribution is a simplex S , so that $Q = S$. If this simplex S is full dimensional, then its volume is given by a determinantal expression, and so, it is rational (see e.g. [GK94] 3.1). Then, the formulas of Table 1 yield the result. If this simplex is not full dimensional, we have $\text{Aff}(S) = \{y \in \mathbb{R}^m \mid \forall j \in J, y_j = q_j\}$, a similar formula holds, ignoring the coordinates of y whose indices are in the set J .

4. *Case of mixtures of distributions.* Trivial reduction to the previous cases. \square

Remark 27. *The conclusion of Theorem 26 does not carry over to the uniform distribution on a general polytope of dimension $k < n$. The condition that $\text{Aff}(Q) = \{y \in \mathbb{R}^m \mid \forall j \in J, y_j = q_j\}$ ensures that the orthogonal projection on $\text{Aff}(Q)$ preserves rationality, which entails that the k -dimensional volume of Q is a rational number. In general, this volume is obtained by applying the Cayley Menger determinant formula (see for example [GK94, 3.6.1]), and it belongs to a quadratic extension of the field of rational numbers. For example, if Δ_d is the canonical simplex $\{\lambda \in \mathbb{R}_+^{d+1} \mid \sum_{i=1}^{d+1} \lambda_i = 1\}$ then $\text{Vol}(\Delta_d) = \frac{\sqrt{d+1}}{d!}$.*

For the Gaussian distribution, \check{c}_S and \check{p}_S can be determined in terms of solid angles (see [Rib06]) arising in Table 1. These coefficients are generally involving the number π and Euler's Γ function, and thus they are irrational.

Corollary 28 (MSLP is polynomial for fixed dimensions). *Consider the problem Eq. (MSLP). Assume that $T, n_2, \dots, n_T, \#(\text{supp } \xi_2), \dots, \#(\text{supp } \xi_T)$ are fixed integers, that $(\mathbf{c}_t, \xi_t)_{2 \leq t \leq T}$ satisfies Assumption 2. Suppose in addition that, for all $\xi \in \text{supp}(\xi_t)$, $p_{t,\xi} := \mathbb{P}[\xi_t = \xi]$ and ξ are rational and that the random variable \mathbf{c}_t conditionally to $\{\xi_t = \xi\}$, denoted by $\mathbf{c}_{t,\xi}$, is of the type considered in Theorem 26.*

Then, Problem (MSLP) can be solved in a time that is polynomial in the input size $\langle c_1 \rangle + \langle \xi_1 \rangle + \sum_{t=2}^T \sum_{\xi \in \text{supp}(\xi_t)} (\langle \mathbf{c}_{t,\xi} \rangle + \langle \xi \rangle + \langle p_{t,\xi} \rangle)$.

Proof. We first show by backward induction that the epigraph $\text{epi}(V_2)$ is well described. The dynamic programming equation Eq. (1) allows us to compute a H -representation of $\text{epi}(V_t)$ from a H -representation of $\text{epi}(V_{t+1})$. Indeed, by Theorem 18, we have

$$V_t(x_{t-1}) = \sum_{\xi \in \text{supp}(\xi_t)} p_{t,\xi} \sum_{N \in \mathcal{N}_{t,\xi}} \check{p}_{t,N|\xi} \min_{x_t \in \mathbb{R}^{n_t}} Q_{t,N|\xi}(x_t, x_{t-1}) \quad , \quad \text{with}$$

$$Q_{t,N|\xi}(x_t, x_{t-1}) := \check{c}_{t,N|\xi}^\top x_t + V_{t+1}(x_t) + \mathbb{I}_{(x_t, x_{t-1}) \in P_t(\xi)} \quad .$$

We then have

$$\begin{aligned} \text{epi}(Q_{t,N|\xi}) &= \left(\text{epi}(x_t \mapsto \check{c}_{t,N|\xi}^\top x_t) + \text{epi}(V_{t+1}) \right) \cap (P_t(\xi) \times \mathbb{R}) \\ \text{epi}(V_t) &= \sum_{\xi \in \text{supp}(\xi_t)} p_{t,\xi} \sum_{N \in \mathcal{N}_{t,\xi}} \check{p}_{t,N|\xi} \pi_{x_{t-1},z}^{x_{t-1},x_t,z}(\text{epi}(Q_{t,N|\xi})) , \end{aligned}$$

recalling that $\pi_{x_{t-1},z}^{x_{t-1},x_t,z}$ denotes the projection mapping $(x_{t-1}, x_t, z) \mapsto (x_{t-1}, z)$. Well described polyhedra are stable under the operations of projection, intersection, and Minkowski sum, see in particular [GLS12, 6.5.18]. It follows that $\text{epi}(V_t)$ is well described. Then, the corollary follows from Lemma 23, Theorem 25 and Theorem 26. \square

6.2 Multistage programming with inexact oracles

We finally consider the situation in which the law of the cost distribution is only known approximately. Hence, we relax the notion of cone-valuation oracle, as follows.

Definition 29 (Weak cone-valuation oracle). *Let $\mathbf{c} \in L(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{R}^m)$ be an integrable cost distribution. We say that \mathbf{c} admits a polynomial time weak cone-valuation oracle, if there exists an oracle which takes as input a rational polyhedral cone N together with a rational number $\varepsilon > 0$, and returns a rational number \tilde{p}_N and a rational vector \tilde{c}_N such that $|\tilde{p}_N - \check{p}_N| \leq \varepsilon$ and $\|\tilde{c}_N - \check{c}_N\| \leq \varepsilon$, in a time that is polynomial in $\langle N \rangle + \langle \varepsilon \rangle$.*

Definition 30 (Weak first-order oracle). *Let f be a rational polyhedral function. We say that f admits a polynomial time weak first-order oracle, if there exists an oracle that takes as input a vector x and either returns a hyperplane separating x from $\text{dom}(f)$ if $x \notin \text{dom}(f)$ or returns a scalar \tilde{f} and a vector \tilde{g} such that $|\tilde{f} - f(x)| \leq \varepsilon$ and $d(\tilde{g}, \partial f(x)) \leq \varepsilon$ if $x \in \text{dom}(f)$, in a time which is polynomial in $\langle x \rangle + \langle \varepsilon \rangle$.*

Remark 31. *In our definition of weak first order oracle, we require that feasibility ($x \in \text{dom}(f)$) be tested exactly, whereas the value and a subgradient of the function are only given approximately. This is suitable to the present setting, in which the main difficulty resides in the approximation of the function (which may take irrational values for relevant cost distributions).*

We now rely on the theory of linear programming with weak separation oracles developed in [GLS12]. Let $C \subset \mathbb{R}^d$ be convex set, for $\varepsilon > 0$, let $S(C, \varepsilon) := \{x \in \mathbb{R}^d \mid \|x - y\| \leq \varepsilon\}$ and $S(C, -\varepsilon) := \{x \in \mathbb{R}^d \mid B(x, \varepsilon) \subset C\}$ where $B(x, \varepsilon)$ denotes the Euclidean ball centered at x of radius ε . A *weak separation oracle* for a convex set $C \subset \mathbb{R}^d$ takes as argument a vector $x \in \mathbb{R}^d$ and a rational number $\varepsilon > 0$, and either asserts that $x \in S(C, \varepsilon)$ or returns a rational vector $\gamma \in \mathbb{R}^d$, of norm one, and a rational scalar δ , such that $\gamma^\top y \leq \gamma^\top x + \varepsilon$ for all $y \in S(C, -\varepsilon)$.

Theorem 32 (Weak cone valuation to weak first-order oracle). *Consider the value functions of problem (MSLP) defined in Eq. (1). Assume that $T, n_2, \dots, n_T, \#\text{supp } \xi_2, \dots, \#\text{supp } \xi_T$ are fixed integers, and that $(\mathbf{c}_t, \xi_t)_{2 \leq t \leq T}$ satisfies Assumption 2. Assume in addition that, every vector $\xi \in \text{supp}(\xi_t)$ has rational entries and that the probabilities $p_{t,\xi} := \mathbb{P}[\xi_t = \xi]$ are rational numbers. Assume finally that the diameters of $\text{dom } V_t$, for $t \geq 2$, are bounded by a rational constant R , and that every random variable \mathbf{c}_t conditionally to $\{\xi_t = \xi\}$, denoted by $\mathbf{c}_{t,\xi}$, admits a polynomial-time weak cone-valuation oracle (see Definition 24).*

Then, for all $t \geq 2$, V_t admits a polynomial time weak first-order oracle.

Proof. The proof is similar to the one of Theorem 25. The main difference is that we need an a priori bound R on the diameter of $\text{dom } V_t$, so that if $d(\tilde{g}, \partial V_t(x)) \leq \varepsilon$, then, using Cauchy-Schwarz inequality, $V_t(y) - V_t(x) \geq \tilde{g} \cdot (y - x) - \varepsilon R$ holds for all $y \in \text{dom } V_t$. Together with and approximation of $V_t(x)$, this allows us to get a weak separation oracle for the epigraph of V_t . \square

Corollary 33 (Approximate (MSLP) is polynomial-time for fixed recourse dimension m). *Consider Problem (MSLP). Assume that $T, n_2, \dots, n_T, \sharp(\text{supp } \xi_2), \dots, \sharp(\text{supp } \xi_T)$ are fixed integers. Assume finally that the diameters of $\text{dom } V_t$, for $t \geq 2$, are bounded by a rational constant R , and that for all $\xi \in \text{supp}(\xi_t)$, the random variable \mathbf{c}_t conditionally to $\{\xi_t = \xi\}$, denoted by $\mathbf{c}_{t,\xi}$, admits a polynomial-time weak cone-valuation oracle.*

Then, there exists an algorithm that either asserts that Problem Eq. (MSLP) is infeasible or find a feasible solution x^ whose cost does not exceed the cost of an optimal solution by more than ε , in polynomial-time in $\langle c_1 \rangle + \langle \xi_1 \rangle + \sum_{t=2}^T \sum_{\xi \in \text{supp}(\xi_t)} (\langle \mathbf{c}_{t,\xi} \rangle + \langle \xi \rangle + \langle p_{t,\xi} \rangle) + \langle R \rangle$.*

Proof. This follows from Theorem 32, using the result analogous to Lemma 23 for weak separation oracles, see [GLS12, 6.5.19]. \square

Finally, we show that every absolutely continuous cost distribution, with a suitable density function, admits a polynomial-time weak cone-valuation oracle.

Definition 34. *We shall say that a density function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is combinatorially tight if:*

1. *there is a polynomial time algorithm which, given a rational number $\varepsilon > 0$, returns a rational number $r > 0$ such that $\int_{\|x\|>r} f(x)dx \leq \varepsilon$.*
2. *there is a polynomial time algorithm, which given a rational vector $x \in \mathbb{R}^n$, and a rational number $\varepsilon > 0$, returns an ε approximation of $f(x)$.*

The terminology is inspired by the notion of tightness from measure theory (analogous to condition 1 in Definition 34).

We shall need a classical result on the numerical approximation of multidimensional integrals, which can be found in [DR84]. The *total variation in the sense of Hardy and Krause*, $\|f\|_{\text{BVHK}}$, of a function f on a n dimensional hypercube is defined in [DR84, Def. p.352]). In particular, if f is of regularity class \mathcal{C}^n , $\|f\|_{\text{BVHK}}$ is finite. The error made when approximating the integral of a function of n variables by its Riemann sum taken on a regular grid with k points is bounded by $(n\|f\|_{\text{BVHK}})/k^{1/n}$, see the theorem on p 352 of [DR84].

Proposition 35. *Suppose that a cost distribution \mathbf{c} admits a density function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$, that is such that the function $(1 + \|\cdot\|)f$ is combinatorially tight and that it has a finite total variation in the sense of Hardy and Krause, bounded by an a priori constant. Suppose that the dimension n is fixed. Then, \mathbf{c} admits a polynomial-time weak cone valuation oracle.*

Proof. Given a rational cone N , we need to approximate the integrals $\int_N f(c)dc$ and $\int_N cf(c)dc$, up to the precision ε . Using the tightness condition, it suffices to approximate the integrals of the same functions restricted to the domain $N_r := N \cap B_\infty(0, r)$, where $B_\infty(0, r)$ denotes the sup-norm ball of radius r , and the encoding length of r is polynomially bounded in the encoding length of ε . We only discuss the approximation of $\int_{N_r} cf(c)dc$ (the case of $\int_{N_r} f(c)dc$ being simpler). We denote by \tilde{c}_{N_r} the approximation of $\int_{N_r} cf(c)dc$ provided by taking the Riemann sum of the function $c \mapsto cf(c)$ over the grid $([-r, r])^n \cap ((r/M)\mathbb{Z})^n$, which has $(2M)^n$ points. Then, setting $g := (1 + \|\cdot\|)f$, it follows from the result [DR84, Th. p 352] recalled above that $\|\int_{N_r} cf(c)dc - \tilde{c}_{N_r}\| \leq n\|g\|_{\text{BVHK}}/(2M)$. Hence, for a fixed dimension n , we can get an ε approximation of $\int_N cf(c)dc$ in a time polynomial in the encoding length of ε . \square

Remark 36. *Proposition 35 and Corollary 33 entail that, under the previous fixed-parameter restrictions (including dimensions of the recourse spaces), the MSLP problem is polynomial-time approximately solvable for a large class of cost distributions. This applies in particular to distributions like Gaussians, which are combinatorially tight. In this case, condition 1 of Definition 34, whereas condition 2 follows from the result of [BB88], implying that the exponential function, restricted to the interval $(-\infty, 0]$, can be approximated in polynomial time.*

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