

Minimum Cost Homomorphisms to Semicomplete Bipartite Digraphs

Gregory Gutin* Arash Rafiey† Anders Yeo‡

Abstract

For digraphs D and H , a mapping $f : V(D) \rightarrow V(H)$ is a homomorphism of D to H if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. If, moreover, each vertex $u \in V(D)$ is associated with costs $c_i(u), i \in V(H)$, then the cost of the homomorphism f is $\sum_{u \in V(D)} c_{f(u)}(u)$. For each fixed digraph H , we have the *minimum cost homomorphism problem for H* . The problem is to decide, for an input graph D with costs $c_i(u), u \in V(D), i \in V(H)$, whether there exists a homomorphism of D to H and, if one exists, to find one of minimum cost. Minimum cost homomorphism problems encompass (or are related to) many well studied optimization problems. We describe a dichotomy of the minimum cost homomorphism problem for semicomplete multipartite digraphs H . This solves an open problem from an earlier paper. To obtain the dichotomy of this paper, we introduce and study a new notion, a k -Min-Max ordering of digraphs.

1 Introduction

Motivation. We consider only directed (undirected) graphs that have neither loops nor multiple arcs (edges). In this paper we solve a problem raised in [5] to find a dichotomy for the computational complexity of minimum cost homomorphism problem (MCH) for semicomplete bipartite digraphs (we define this problem below). In fact, our result leads to a complete dichotomy for the computational complexity of MCH for semicomplete k -partite digraphs ($k \geq 2$) as a (much simpler) dichotomy for the case $k \geq 3$ was obtained in [5] (see also Section 5). Our result uses and significantly extends a dichotomy for the computational complexity of MCH for bipartite undirected graphs obtained in [3].

*Corresponding author. Department of Computer Science, Royal Holloway University of London, Egham, Surrey TW20 OEX, UK, gutin@cs.rhul.ac.uk

†School of Computing Science, Simon Fraser University, Burnaby, B.C., Canada, V5A 1S6, arafeyh@cs.sfu.ca

‡Department of Computer Science, Royal Holloway University of London, Egham, Surrey TW20 OEX, UK, anders@cs.rhul.ac.uk

In our previous papers we used properties of an important notion of Min-Max ordering of digraphs. To obtain the dichotomy of this paper, we introduce and study a new notion, a k -Min-Max ordering of digraphs. We believe that properties of this notion and, in particular, Theorem 2.2 can be used to obtain further results on MCH and its special cases (see below).

The minimum cost homomorphism problem was introduced in [6], where it was motivated by a real-world problem in defence logistics. We believe it offers a practical and natural model for optimization of weighted homomorphisms. MCH's special cases include the well-known list homomorphism problem [8, 10] and the general optimum cost chromatic partition problem, which has been intensively studied [7, 11, 12], and has a number of applications, [14, 15].

Minimum cost homomorphisms. For directed or undirected graphs G and H , a mapping $f : V(G) \rightarrow V(H)$ is a *homomorphism of G to H* if $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. Recent treatments of homomorphisms in directed and undirected graphs can be found in [8, 10]. Let H be a fixed directed or undirected graph. The *homomorphism problem* for H asks whether a directed or undirected input graph G admits a homomorphism to H . The *list homomorphism problem* for H asks whether a directed or undirected input graph G with lists (sets) $L_u \subseteq V(H), u \in V(G)$ admits a homomorphism f to H in which $f(u) \in L_u$ for each $u \in V(G)$.

Suppose G and H are directed (or undirected) graphs, and $c_i(u), u \in V(G), i \in V(H)$ are nonnegative *costs*. The *cost of a homomorphism f of G to H* is $\sum_{u \in V(G)} c_{f(u)}(u)$. If H is fixed, the *minimum cost homomorphism problem*, $\text{MinHOM}(H)$, for H is the following optimization problem. Given an input graph G , together with costs $c_i(u), u \in V(G), i \in V(H)$, we wish to find a minimum cost homomorphism of G to H , or state that none exists.

A bipartite digraph is *semicomplete* if there is at least one arc between every two vertices belonging to different partite sets. In this paper, we study the *minimum cost homomorphism problem for semicomplete bipartite digraphs*, i.e., $\text{MinHOM}(H)$ when H is a *semicomplete bipartite digraph*. Observe that MCH for semicomplete bipartite digraphs extends MCH for bipartite undirected graphs. Indeed, let B be a semicomplete bipartite digraph with partite sets U, V and arc set $A(B) = A_1 \cup A_2$, where $A_1 = \{uv : u \in U, v \in V\}$ and $A_2 \subseteq \{vu : v \in V, u \in U\}$. Let B' be a bipartite graph with partite sets U, V and edge set $E(B') = \{uv : vu \in A_2\}$. Notice that $\text{MinHOMP}(B)$ is equivalent to $\text{MinHOMP}(B')$.

Min-Max ordering. Let H be a digraph. We say that an ordering v_1, v_2, \dots, v_p of $V(H)$ is a *Min-Max ordering of H* if $v_i v_r, v_j v_s \in A(H)$ implies $v_{\min\{i,j\}} v_{\min\{s,r\}} \in A(H)$ and $v_{\max\{i,j\}} v_{\max\{s,r\}} \in A(H)$. One can easily see that v_1, v_2, \dots, v_p of $V(H)$ is a Min-Max ordering of H if $i < j, s < r$ and $v_i v_r, v_j v_s \in A(H)$, then $v_i v_s \in A(H)$ and $v_j v_r \in A(H)$. We can define a Min-Max ordering for a bipartite undirected graph G with partite sets

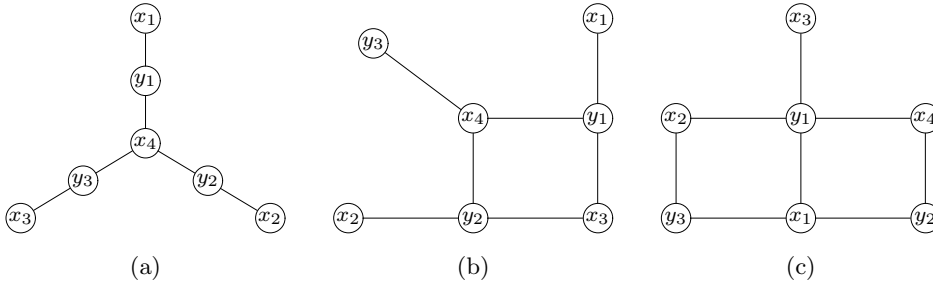


Figure 1: A bipartite claw (a), a bipartite net (b) and a bipartite tent (c).

V and U as follows: We orient all edges from V to U and apply the above definition for digraphs. Importance of Min-Max ordering for $\text{MinHOM}(H)$ is indicated in the following two theorems.

Theorem 1.1 [4] *Let a digraph H have a Min-Max ordering. Then $\text{MinHOM}(H)$ is polynomial-time solvable.*

A bipartite graph H with vertices $x_1, x_2, x_3, x_4, y_1, y_2, y_3$ is called

a *bipartite claw* if its edge set $E(H) = \{x_4y_1, y_1x_1, x_4y_2, y_2x_2, x_4y_3, y_3x_3\}$;

a *bipartite net* if its edge set $E(H) = \{x_1y_1, y_1x_3, y_1x_4, x_3y_2, x_4y_2, y_2x_2, y_3x_4\}$;

a *bipartite tent* if its edge set $E(H) = \{x_1y_1, y_1x_3, y_1x_4, x_3y_2, x_4y_2, y_2x_2, y_3x_4\}$.

See Figure 1.

Theorem 1.2 [3] *Let H be an undirected bipartite graph. If H contains a cycle C_{2k} , $k \geq 3$ or a bipartite claw or a bipartite net or a bipartite tent as an induced subgraph, then $\text{MinHOM}(H)$ is NP-hard.*

Assume that $P \neq \text{NP}$. Then the following three assertions are equivalent:

(i) H has a Min-Max ordering;

(ii) $\text{MinHOM}(H)$ is polynomial time solvable;

(iii) H does not contain a cycle C_{2k} , $k \geq 3$, a bipartite claw, a bipartite net, or a bipartite tent as an induced subgraph.

Additional terminology and notation. For a graph H , $V(H)$ and $E(H)$ denote its vertex and edge sets, respectively. For a digraph H , $V(H)$ and $A(H)$ denote its vertex

and arc sets, respectively. For a pair X, Y of vertex sets of a digraph H , $(X, Y)_H$ denotes the set of all arcs of the form xy , where $x \in X, y \in Y$. We omit the subscript when it is clear from the context. Also, $X \times Y = \{xy : x \in X, y \in Y\}$. For a set $X \subseteq V(H)$, let $N^+(X) = \{y : \exists x \in X \text{ with } xy \in A(H)\}$ and $N^-(X) = \{y : \exists x \in X \text{ with } yx \in A(H)\}$.

If xy is an arc of a digraph H , we will say that x *dominates* y , y is *dominated by* x , y is an *out-neighbor* of x , and x is an *in-neighbor* of y . We also denote it by $x \rightarrow y$. For disjoint sets $X, Y \subseteq V(H)$, $X \rightarrow Y$ means that $x \rightarrow y$ for each $x \in X$ and $y \in Y$.

An *extension* of a digraph G is a digraph D obtained from G by replacing each vertex u of G by a set of independent vertices $u_1, u_2, \dots, u_{n(u)}$ such that for a pair u, v of vertices in G , $u_i \rightarrow v_j$ in D if and only if $u \rightarrow v$ in G .

For a bipartite digraph $H = (V, U; A)$, where V and U are its partite sets, H^{\rightarrow} is the subdigraph induced by all arcs directed from V to U , H^{\leftarrow} is the subdigraph induced by all arcs directed from U to V , and H^{\leftrightarrow} is the subdigraph induced by all 2-cycles of H , i.e., by the set $\{xy : xy \in A, yx \in A\}$. The *converse* of H is the digraph obtained from H by replacing every arc xy with the arc yx .

We denote a directed cycle with p vertices by \vec{C}_p . For a set X of vertices of a digraph H , $D[X]$ denotes the subdigraph of H induced by X . For a digraph H , $UN(H)$ denotes the *underlying graph* of H , i.e., an undirected graph obtained from H by disregarding all orientations and deleting multiple edges.

A digraph D is *strong* (or, *strongly connected*) if there is a directed path from x to y and a directed path from y to x for every pair x, y of vertices of D .

Forbidden family. Let us introduced five special digraphs for which, as we will see later, the minimum homomorphism problem is NP-hard. The digraph C'_4 has vertex set $\{x_1, x_2, y_1, y_2\}$ and arc set $\{x_1y_1, y_1x_2, x_2y_2, y_2x_1, y_1x_1\}$. The digraph C''_4 has the same vertex set, but its arc set is $A(C'_4) \cup \{x_2y_1\}$. The digraph H^* has vertex set $\{x_1, x_2, y_1, y_2, y_3\}$ and arc set

$$\{x_1y_1, y_1x_2, x_2y_2, y_2x_1, x_1y_3, x_2y_3\}.$$

Let N_1 be a digraph with $V(N_1) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and

$$A(N_1) = \{x_1y_1, y_1x_1, x_2y_2, y_2x_2, x_3y_3, y_3x_3, y_1x_2, y_1x_3, x_1y_2, x_1y_3, x_3y_2, x_2y_3\}.$$

Let N_2 be a digraph with $V(N_2) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and

$$A(N_2) = \{x_1y_1, x_2y_2, y_2x_2, x_3y_3, y_3x_3, y_1x_2, y_1x_3, x_1y_2, x_1y_3, x_3y_2, x_2y_3\}.$$

A digraph H belongs to the family \mathcal{HFORB} if H or its converse is isomorphic to one of the five digraphs above or $UN(H^s)$ is isomorphic to bipartite claw, bipartite net, bipartite tent or even cycle with at least 6 vertices, where $s \in \{\rightarrow, \leftarrow, \leftrightarrow\}$.

k -Min-Max ordering. A collection V_1, V_2, \dots, V_k of subsets of a set V is called a *k -partition* of V if $V = V_1 \cup V_2 \cup \dots \cup V_k$, $V_i \cap V_j = \emptyset$ provided $i \neq j$.

Definition 1.3 Let $H = (V, A)$ be a digraph and let $k \geq 2$ be an integer. We say that H has a k -Min-Max ordering if there is a k -partition of V into subsets V_1, V_2, \dots, V_k and there is an ordering $v_1^i, v_2^i, \dots, v_{\ell(i)}^i$ of V_i for each i such that

- (i) Every arc of H is an (V_i, V_{i+1}) -arc for some $i \in \{1, 2, \dots, k\}$,
- (ii) $v_1^i, v_2^i, \dots, v_{\ell(i)}^i, v_1^{i+1}, v_2^{i+1}, \dots, v_{\ell(i+1)}^{i+1}$ is a Min-Max ordering of the subdigraph $H[V_i \cup V_{i+1}]$ for all $i \in \{1, 2, \dots, k\}$,

where all indices $i + 1$ are taken modulo k .

Note that if H is a strong digraph in which the greatest common divisor of all cycle lengths is k , then $V(H)$ has a k -partition, $k \geq 2$, satisfying (i) (see Theorem 10.5.1 in [1]). A simple example of a digraph having a k -Min-Max ordering is an extension of \vec{C}_k .

Dichotomy and paper organization. The main result of this paper is the following:

Theorem 1.4 Let H be an semicomplete bipartite digraph. If H contains a digraph from \mathcal{HFORB} as an induced subdigraph, then $\text{MinHOM}(H)$ is NP-hard.

Assume that $P \neq NP$. Then the following three assertions are equivalent:

- (i) $\text{MinHOM}(H)$ is polynomial time solvable;
- (ii) H does not contain a digraph from \mathcal{HFORB} as an induced subdigraph;
- (iii) Each component of H has a k -Min-Max ordering for $k = 2$ or 4 .

Theorem 1.4 follows from Corollaries 3.5 and 4.6.

The rest of the paper is organized as follows. In Section 2, we study properties of k -Min-Max orderings. In Section 3, we prove polynomial cases of $\text{MinHOM}(H)$ when H is a semicomplete bipartite digraph. In Section 4, we establish NP-hard cases of the problem. In Section 5 we formulate a dichotomy for the computational complexity of $\text{MinHOM}(H)$ when H is a semicomplete multipartite digraph. Section 6 provides a short discussion of further research.

2 Properties of k -Min-Max Orderings

Digraphs having k -Min-Max ordering have a very special structure as described in the following lemma.

Lemma 2.1 If a strong digraph $H = (V, A)$ has a k -Min-Max ordering as described in Definition 1.3, then we can define $0 \leq L(i, j) < R(i, j) \leq \ell(j) + 1$ for all i and j such that

(a): $N^+(v_i^j) = \{v_{L(i,j)+1}^{j+1}, v_{L(i,j)+2}^{j+1}, \dots, v_{R(i,j)-1}^{j+1}\}$, where all superscripts are taken modulo k ;

(b): For all j and $i < i'$ we have $R(i, j) \leq R(i', j)$ and $L(i, j) \leq L(i', j)$.

Proof: Suppose that $v_i^j v_{m-1}^{j+1}, v_i^j v_{m+1}^{j+1} \in A$, but $v_i^j v_m^{j+1} \notin A$. Since H is strong, there is an arc $v_i^j v_m^{j+1}$ in H . By the definition of a k -Min-Max ordering, $v_i^j v_m^{j+1} \in A$, a contradiction that proves (a). Similarly, one can show (b). \diamond

The construction used in the following theorem was inspired by somewhat similar constructions in [13] and [2].

Theorem 2.2 *If a digraph H has a k -Min-Max ordering, then $\text{MinHOM}(H)$ is polynomial-time solvable.*

Proof: Let V_1, V_2, \dots, V_k be defined as in Definition 1.3. Let D be an input digraph. Assume that there is homomorphism f of D to H . Let G_1, G_2, \dots, G_k be a k -partition of $V(D)$ such that $f(G_j) \subseteq V_j$ for each $j \in \{1, 2, \dots, k\}$. Observe that all arcs in D are (G_j, G_{j+1}) -arcs, where all indices are taken modulo k and $j \in \{1, 2, 3, \dots, k\}$.

We will now show how to find a minimum cost homomorphism of D to H , where the vertices of G_j are mapped to V_j for all $j = 1, 2, \dots, k$. We will build a directed graph \mathcal{L} with vertex set $\cup_{j=1}^k (G_j \times V_j)$ together with two other vertices, denoted by s and t . We will also denote t by $(x, v_{\ell(j)+1}^j)$ for every $j \in \{1, 2, \dots, k\}$. The weighted arcs of \mathcal{L} are as follows, where M is any constant greater than the cost of a minimum cost homomorphism of D to H .

- An arc from s to (x, v_1^j) , of weight ∞ , for each $x \in G_j$.
- An arc from (x, v_i^j) to (x, v_{i+1}^j) , of weight $c_i(x) + M$, for each $x \in G_j$ and $i \in \{1, 2, \dots, \ell(j)\}$. Recall that when $i = \ell(j)$ the arc enters t .
- an arc from (x, v_i^j) to $(y, v_{L(i,j)+1}^{j+1})$ and an arc from $(y, v_{R(i,j)}^{j+1})$ to (x, v_{i+1}^j) for every $xy \in A(D)$ with $x \in G_j$ and every $i = 1, 2, \dots, \ell(j)$. Furthermore the weight of these arcs are ∞ .

A *cut* in \mathcal{L} is a partition of the vertices into two sets S and T such that $s \in S$ and $t \in T$ and the weight of a cut is the sum of weights of all arcs going from a vertex of S to a vertex of T . We will show that the minimum weight cut in \mathcal{L} has weight equal to the minimum cost homomorphism of D to H plus $|V(D)|M$.

Let f be a minimum cost homomorphism of D to H , and assume that $f(x) = v_{a(x)}^j$ for each $x \in G_j$ and for all $j = 1, 2, \dots, k$. Define a cut in D as follows: $S = \{(x, v_i^j) :$

$i \leq a(x)$, $j = 1, 2, \dots, k\} \cup \{s\}$ and $T = V(L) - S$. Note that the arcs from $(x, v_{a(x)}^j)$ to $(x, v_{a(x)+1}^j)$ belong to the cut and contribute $c_{f(x)}(x) + M$ to the weight of the cut. We will now show that there are no arcs of infinite weight in the cut, which would imply that the weight of S is exactly the cost of a minimum cost homomorphism from D to H plus $|V(D)|M$.

Clearly no arc out of s belongs to the cut S . Assume for the sake of contradiction that the arc (x, v_i^j) to $(y, v_{L(i,j)+1}^{j+1})$ belongs to the cut S for some $xy \in A(D)$ with $x \in G_j$. This implies that $a(x) \geq i$ (as $(x, v_i^j) \in S$) and $a(y) < L(i, j) + 1$ (as $(y, v_{L(i,j)+1}^{j+1}) \notin S$). By Lemma 2.1 (b), this implies that $a(y) \leq L(i, j) \leq L(a(x), j)$. Thus, there is no arc from $v_{a(x)}^j$ to $v_{a(y)}^{j+1}$ in H , by the definition of $L(i, j)$. This is a contradiction to f being a homomorphism.

Now assume for the sake of contradiction that the arc $(y, v_{R(i,j)}^{j+1})$ to (x, v_{i+1}^j) belongs to the cut S for some $xy \in A(D)$ with $x \in G_j$. This implies that $a(x) < i + 1$ (as $(x, v_{i+1}^j) \notin S$) and $a(y) \geq R(i, j)$ (as $(y, v_{R(i,j)}^{j+1}) \in S$). By Lemma 2.1 (b), this implies that $a(y) \geq R(i, j) \geq R(a(x), j)$. Thus, there is no arc from $v_{a(x)}^j$ to $v_{a(y)}^{j+1}$ in H , by the definition of $R(i, j)$. This is a contradiction to f being a homomorphism. We have now proved that the cut S has the stated weight.

For the sake of contradiction assume that there exists a cut, S' , in \mathcal{L} of smaller weight than S . As the weight of S is less than $M + |V(D)|M$ we note that the cut S' contains exactly one arc of the form $(x, v_i^j)(x, v_{i+1}^j)$ for each $x \in V(D)$. Therefore we may define a mapping, f' , from $V(D)$ to $V(H)$ by letting $f'(x) = v_i^j$ if and only if $(x, v_i^j) \in S'$ and $(x, v_{i+1}^j) \notin S'$. We will now show that f' is a homomorphism of D to H of smaller cost than f , a contradiction. This would imply that S is a minimum weight cut, and we would be done.

Note that if f' is a homomorphism of D to H , then it has smaller cost than f , as S' is a cut of smaller weight than S . Let xy be any arc in D , and assume without loss of generality that $x \in G_j$. Let i_x and i_y be defined such that $(x, v_{i_x}^j) \in S'$ and $(x, v_{i_x+1}^j) \notin S'$ (i.e., $f'(x) = v_{i_x}^j$) and $(y, v_{i_y}^{j+1}) \in S'$ and $(y, v_{i_y+1}^{j+1}) \notin S'$ (i.e., $f'(y) = v_{i_y}^{j+1}$). As the arc $(x, v_{i_x}^j)(y, v_{L(i_x,j)+1}^{j+1})$ is not in the cut, we must have $(y, v_{L(i_x,j)+1}^{j+1}) \in S$, which implies that $i_y \geq L(i_x, j) + 1$. Furthermore as the arc $(y, v_{R(i_x,j)}^{j+1})(x, v_{i_x+1}^j)$ is not in the cut, we must have $(y, v_{R(i_x,j)}^{j+1}) \notin S$, which implies that $i_y < R(i_x, j)$. We have now shown that $L(i_x, j) + 1 \leq i_y < R(i_x, j)$, which by the definition of the functions L and R implies that $v_{i_x}^j v_{i_y}^{j+1}$ is an arc in H . Therefore f' is a homomorphism. \diamond

3 Polynomial Cases

We start from a special case which is of importance when H contains no induced \vec{C}_4 .

Lemma 3.1 *Let $H = (V, U; A)$ be a semicomplete bipartite digraph, which does not contain an induced subdigraph belonging to \mathcal{HFORB} or an induced directed 4-cycle. Suppose for every $v, v' \in V$ we have $N^+(v) \subseteq N^+(v')$ or $N^+(v') \subseteq N^+(v)$. Then H has a 2-Min-Max ordering.*

Proof: We say that vertices $v_i, v_j \in V$ are *similar* if $N^+(v_i) = N^+(v_j)$. Consider similarity classes V_1, V_2, \dots, V_s of V . Moreover assume that $N^+(V_i) \subset N^+(V_j)$ for $i < j$. Set $U_1 = N^+(V_1)$ and $U_i = N^+(V_i) - \cup_{j=1}^{i-1} U_j$ for each $i > 1$. If $s = 1$ then $UN(H^{\rightarrow})$ is a complete bipartite graph and a Min-Max ordering of H^{\leftarrow} (which exists by Theorem 1.2) is a 2-Min-Max ordering of H . Assume $s > 1$. We prove the following two claims:

- (1) Let $u_i \in U_i$, $v_j \in V_j$, $j > i$, and $u_i v_j \in A$. Then $U_r \rightarrow v_j$ for each $r > i$ and $u_i \rightarrow V_t$ for each $t < j$.

Proof of (1): By the definition of U_i and V_i , if $r > j$ then $U_r \rightarrow v_j$. Now suppose that $i < r \leq j$ and $u_r v_j \notin A$ for some $u_r \in U_r$. Let $v_i \in V_i$ be arbitrary, and note that $H[\{u_i, v_i, u_r, v_j\}]$ is isomorphic to C'_4 or C''_4 , a contradiction. So $u_r v_j \in A$.

By the definition of U_i and V_i , if $t < i$ then $u_i \rightarrow V_t$. Now suppose that $i \leq t < j$ and that $u_i v_t \notin A$, for some $v_t \in V_t$. Let $u_j \in U_j$ be arbitrary, and note that $H[\{u_i, v_t, u_j, v_j\}]$ is isomorphic to C'_4 or C''_4 , a contradiction.

- (2) If $u, u' \in U$ then $N^+(u) \subseteq N^+(u')$ or $N^+(u') \subseteq N^+(u)$.

Proof of (2): Assume that this is not the case, and there exist $v \in N^+(u) - N^+(u')$ and $v' \in N^+(u') - N^+(u)$. This implies that $u \neq u'$, $v \neq v'$, $uv, u'v' \in A$ and $uv', u'v \notin A$. If $u \in U_i$ and $u' \in U_j$ and $i < j$, then by (1) we note that $u'v \in A$ a contradiction. So for some i we must have $\{u, u'\} \subseteq U_i$. Analogously, if $v \in V_a$ and $v' \in V_b$ and $a < b$, then by (1) we note that $u'v \in A$, a contradiction. So for some j we must have $\{v, v'\} \subseteq V_j$. If $i > j$ then $U_i \rightarrow V_j$, which is a contradiction, so we must have $i \leq j$.

If $i < j$ then let $u'' \in U_j$ and $v'' \in V_i$ be arbitrary. Note that by (1) we must have the arcs $u''v, u''v', uv'', u'v''$ in H . By the construction of the sets U_i, U_j, V_i, V_j we now note that the underlying graph of $H[\{u, u', u'', v, v', v''\}]^{\leftrightarrow}$ is the 6-cycle $v''uvu''v'u'v''$, a contradiction.

Therefore we must have $i = j$. First assume that $i < s$. Now consider $v_s \in V_s$ and $u_s \in U_s$. By (1) there is no arc from $\{u, u'\}$ to v_s . However $H[\{u, u', v, v', v_s, u_s\}]$ is either N_1 or N_2 , a contradiction.

Similarly for the case $i = s$ we derive a contradiction.

Now consider an ordering (u_1, u_2, \dots, u_a) of the vertices in U and an ordering (v_1, v_2, \dots, v_b) of the vertices in V , defined as follows. If $i < j$ then $d^+(v_i) \leq d^+(v_j)$ and if $d^+(v_i) = d^+(v_j)$ then $d^-(v_i) \geq d^-(v_j)$. Furthermore when $i < j$ then $d^+(u_i) \leq d^+(u_j)$ and if $d^+(u_i) = d^+(u_j)$ then $d^-(u_i) \geq d^-(u_j)$. Note that the ordering (v_1, v_2, \dots, v_b) first contains vertices from V_1 then from V_2 , etc.

We will now show that for every $i \in \{1, 2, \dots, b\}$ there exists an integer α_i such that $N^+(u_i) = \{v_1, v_2, \dots, v_{\alpha_i}\}$. Suppose this is not the case. Then there exists an arc $u_i v_j$ in H , such that $u_i v_{j-1}$ is not an arc in H . Thus, both v_j and v_{j-1} belong to some V_k , as otherwise we have a contradiction to (1). This implies that $d^+(v_j) = d^+(v_{j-1})$ and $d^-(v_{j-1}) \geq d^-(v_j)$. Note that every vertex $u \in N^-(v_{j-1})$ has $N^+(u_i) \subseteq N^+(u)$, by (2). Therefore $u \rightarrow v_j$. However this implies that $d^-(v_{j-1}) < d^-(v_j)$ (as $u_i \rightarrow v_j$ but u_i does not dominate v_{j-1}), a contradiction.

Using the fact that $N^+(u_i) = \{v_1, v_2, \dots, v_{\alpha_i}\}$ for each $i \in \{1, 2, \dots, b\}$ and that $\alpha_i \geq \alpha_{i-1}$ for each $i \in \{2, 3, \dots, b\}$ (as $d^+(u_i) \geq d^+(u_{i-1})$) and the similar relations for the vertices of V , we can readily conclude that H has a 2-Min-Max ordering. \diamond

The *distance* $\text{dist}(x, y)$ between a pair x, y of vertices in an undirected graph G is the length of the shortest path between x and y . The *diameter* of G is the maximal distance between a pair of vertices in G .

The following theorem shows when $\text{MinHOM}(H)$ is polynomial time solvable if H is strong and does not contain \vec{C}_4 as an induced subdigraph.

Theorem 3.2 *Let H be a strongly connected semicomplete bipartite digraph. Assume that H does not contain a digraph from \mathcal{HFORB} or \vec{C}_4 as an induced subdigraph. Then H has a 2-Min-Max ordering and $\text{MinHOM}(H)$ is polynomial time solvable.*

Furthermore, either $UN(H^{\rightarrow})$ or $UN(H^{\leftarrow})$ are complete bipartite graphs, or the following holds. For every pair u, u' of distinct vertices of U we have $N^+(u) \subseteq N^+(u')$ or $N^+(u') \subseteq N^+(u)$ and for every pair v, v' of distinct vertices of V we have $N^+(v) \subseteq N^+(v')$ or $N^+(v') \subseteq N^+(v)$.

Proof: By Theorem 2.2, to prove the first part part of this theorem (before ‘Furthermore’), it suffices to show that H has a 2-Min-Max ordering. Let V and U be partite sets of H . Denote $H_1 = H^{\rightarrow}$ and $H_2 = H^{\leftarrow}$. It follows from Theorem 1.2 that $UN(H_1)$ and $UN(H_2)$ have Min-Max orderings and so do H_1 and H_2 .

Let d_i be the diameter of $UN(H_i)$, $i = 1, 2$. Observe that if $UN(H_1)$ ($UN(H_2)$) is a complete bipartite graph, then a Min-Max ordering of H_2 (H_1) is a 2-Min-Max ordering of H . Therefore, $\text{MinHOM}(H)$ is polynomial time solvable by Theorem 2.2. Notice that

$UN(H_i)$ is complete bipartite if and only if $d_i = 2$. Thus, we may assume that both $d_1 \geq 3$ and $d_2 \geq 3$.

We consider the following cases for the value of $d_1 \geq 3$.

Case 1: $d_1 > 4$.

We will show that $UN(H_2)$ is a complete bipartite graph or, equivalently, $d_2 = 2$. Assume that d_1 is odd, as the case of d_1 even can be considered similarly. Let $P = v_1 u_1 v_2 u_2 \dots v_{k-1} u_{k-1} v_k u_k$ be a shortest path of length d_1 between v_1 and u_k in $UN(H_1)$. Let $\hat{U} = \{u_1, u_2, \dots, u_k\}$ and $\hat{V} = \{v_1, v_2, \dots, v_k\}$. We will first prove that $\hat{U} \rightarrow \hat{V}$. Since P is a shortest path, we have $u_i \rightarrow v_j$ for each $u_i \in \hat{U}$ and $v_j \in \hat{V}$ provided $j \notin \{i, i+1\}$. Thus, it sufficient to prove

$$u_i \rightarrow v_i \quad (i = 1, 2, \dots, k) \text{ and } u_i \rightarrow v_{i+1} \quad (i = 1, 2, \dots, k-1) \quad (1)$$

Consider the subdigraph H' of H induced by four vertices $v_i, u_i, v_{i+2}, u_{i+1}$, where $i \in \{1, 2, \dots, k-2\}$. By the definition of P (including the fact that P is a shortest path), we have $v_i u_i, v_{i+2} u_{i+1}, u_{i+1} v_i, u_i v_{i+2} \in A(H)$, but $v_i u_{i+1}, v_{i+2} u_i \notin A(H)$. Since H' is not isomorphic to either \tilde{C}_4 or C'_4 , we have $u_i v_i, u_{i+1} v_{i+2} \in A(H)$. This proves that $u_i \rightarrow v_i$ provided $i = 1, 2, \dots, k-2$ and $u_i \rightarrow v_{i+1}$ provided $i = 2, 3, \dots, k-1$. Thus, to prove (1) it remains to show that

$$u_{k-1} \rightarrow v_{k-1}, \quad u_k \rightarrow v_k, \quad u_1 \rightarrow v_2 \quad (2)$$

Consider the subdigraph H'' of H induced by four vertices $v_{k-1}, u_{k-1}, v_k, u_k$. By the definition of P , we have $v_{k-1} u_{k-1}, v_k u_k, v_k u_{k-1}, u_k v_{k-1} \in A(H)$, but $v_{k-1} u_k \notin A(H)$. We have proved that $u_{k-1} \rightarrow v_k$. Since H'' is not isomorphic to C'_4 or C''_4 , we have $u_{k-1} v_{k-1}, u_k v_k \in A(H)$.

Consider $H[\{v_2, v_3, u_1, u_2\}]$. By the definition of P , we have $v_2 u_1, v_2 u_2, v_3 u_2, u_1 v_3 \in A(H)$, but $v_3 u_1 \notin A(H)$. We have proved that $u_2 \rightarrow v_2$. Since $H[\{v_2, v_3, u_1, u_2\}]$ is not isomorphic to C'_4 or C''_4 , we have $u_1 v_2, u_2 v_3 \in A(H)$. This implies that (2) and, thus, (1) has been proved.

Consider vertex $u \in U - \hat{U}$; we will show that $u \rightarrow \{v_1, v_2, \dots, v_k\}$. Suppose this is not true. Let j be the smallest index such that $u v_j \notin A(H)$. We have $v_j u \in A(H)$. Suppose $j > 1$. Since $H[\{u, v_1, u_1, v_j\}]$ is not isomorphic to C'_4 or C''_4 , we have $v_j u_1, v_1 u \in A(H)$. Since P is a shortest path, we have $j = 2$ as otherwise $v_1 u v_j u_{j+1} \dots v_j u_k$ is shorter than P . We have $v_3 u \notin A(H)$ as otherwise $v_1 u v_3 u_3 \dots v_k u_k$ is shorter than P . However, C'_4 is isomorphic to $H[\{v_2, u, v_3, u_3\}]$, a contradiction.

Now assume that $j = 1$. We have $v_3 u \notin A(H)$ as otherwise we have a shorter path. Since H is semicomplete bipartite, we have $u v_3 \in A(H)$. However $H[\{v_1, u, v_3, u_2\}] \cong C'_4$, a contradiction.

Analogously we can prove that $\hat{U} \rightarrow v$ for every $v \in V - \hat{V}$. Consider $u \in U - \hat{U}$, $v \in V - \hat{V}$. We show that $uv \in A(H)$. Suppose this is not true. We have $vu \in A(H)$. Since $H[\{v, u, v_1, u_1\}]$ is not isomorphic to \vec{C}_4 , C'_4 or C''_4 we have $v_1u, vu_1 \in A(H)$. Since $H[\{v, u, v_k, u_k\}]$ is not isomorphic to C'_4 or C''_4 we have $v_ku, vu_k \in A(H)$. But now $\text{dist}(v_1, u_k) = 3$ in $UN(H_1)$, a contradiction.

Case 2: $d_1 = 4$.

We will show that again $UN(H_2)$ is a complete bipartite graph. Assume that $v_1u'_1v'_2u'_2v'_3$ is a shortest path between a pair $v_1 \in V$ and $v'_3 \in U$ in $UN(H_1)$. Let $U_1 = N^+(v_1)$, $V_2 = N^-(U_1) - \{v_1\}$, $U_2 = N^+(V_2) - U_1$ and $V_3 = N^-(U_2) - V_2$. By the definitions, $V = \{v_1\} \cup V_2 \cup V_3$ and $U = U_1 \cup U_2$. Observe also that $(U_1, V_3) = U_1 \times V_3$, $(V_3, U_1) = \emptyset$, $(U_2, \{v_1\}) = U_2 \times \{v_1\}$ and $(\{v_1\}, U_2) = \emptyset$. Let $u_1 \in U_1$, $u_2 \in U_2$, $v_2 \in V_2$ and $v_3 \in V_3$ be arbitrary. If $u_2v_3 \notin A(H)$, then $v_3u_2 \in A(H)$, and now $H[\{u_1, v_3, u_2, v_1\}]$ is either \vec{C}_4 or C'_4 , a contradiction. Therefore, we have $u_2v_3 \in A(H)$ and consequently $(U_2, V_3) = U_2 \times V_3$. Consider v_3, u_2 where $v_3u_2 \in A(H)$. Since $H[\{u_1, v_3, u_2, v_1\}]$ is not C'_4 or \vec{C}_4 , we conclude that $u_1v_1 \in A(H)$ and consequently $(U_1, \{v_1\}) = U_1 \times \{v_1\}$.

Note that we have already proved that the underlying graph of $H_2[U_1 \cup U_2 \cup V_3 \cup \{v_1\}]$ is a complete bipartite graph. Suppose $u_1v_2 \in A(H)$. Then $u_2v_2 \in A(H)$ as otherwise $H[\{u_1, v_1, u_2, v_2\}]$ is isomorphic to C'_4 or C''_4 , a contradiction. Therefore, if $u_1v_2 \in A(H)$, then $(U_2, \{v_2\}) = U_2 \times \{v_2\}$. Thus, to show that $(U_2, V_2) = U_2 \times V_2$ it suffices to prove that every vertex in V_2 has an in-neighbor in U_1 . Suppose this is not true, and let X_2 be the set of all vertices in V_2 that does not have an in-neighbor in U_1 . Let Y_2 be the set of all vertices in U_2 that have an arc into X_2 . As H is strong some vertex in H must have an arc into X_2 , which implies that $Y_2 \neq \emptyset$.

If there is an arc v_3y_2 from V_3 to Y_2 , then let x_2 be an out-neighbor of y_2 in X_2 and let $u_1 \in U_1$ be arbitrary. However this is a contradiction to $H[\{v_3, y_2, x_2, u_1\}]$ not being isomorphic to C'_4 and C''_4 , which implies that there is no arc from V_3 to Y_2 .

As H is strong and there is no arc from V_3 into U_1 or Y_2 , there must be an arc, say v_3u_2 , from V_3 into $U_2 - Y_2$. As H is strong there must also be an arc from $V(H) - X_2 - Y_2$ into $X_2 \cup Y_2$. By the above this arc, say v_2y_2 , must be from $V_2 - X_2$ to Y_2 . As y_2 belongs to Y_2 there must be a vertex, say $x_2 \in X_2$, such that $y_2x_2 \in A(H)$. As $v_2 \notin X_2$ we note that there is a vertex, say $u_1 \in U_1$, such that $u_1v_2 \in A(H)$. As $u_1x_2 \notin A(H)$ and $H[\{v_2, x_2, u_1, y_2\}]$ is not isomorphic to C'_4 and C''_4 , we note that $x_2y_2, v_2u_1 \in A(H)$. If $u_2v_2 \notin A(H)$, then $H[\{v_1, v_2, u_1, u_2\}]$ is isomorphic to C'_4 and C''_4 (as $v_1u_2 \notin A(H)$), a contradiction. As $u_2v_2 \in A(H)$ and $H[\{v_2, v_3, u_2, y_2\}]$ is not isomorphic to C'_4 and C''_4 we note that $y_2v_2 \in A(H)$ (as $v_3y_2 \notin A(H)$). As $H[\{v_2, x_2, u_2, y_2\}]$ is not isomorphic to C'_4 and C''_4 we note that $v_2u_2 \in A(H)$ (as $u_2x_2 \notin A(H)$). However, the underlying graph of $H[v_1, u_1, v_2, v_3, u''_2, v''_2, u_2] \leftarrow$ is now a bipartite claw, with edges $\{v_2u_1, v_2y_2, v_2u_2, u_1v_1, y_2x_2, u_2v_3\}$, a contradiction. Therefore $U_2 \rightarrow V_2$.

We show that $u_1v_2 \in A(H)$ for every $u_1 \in U_1$ and $v_2 \in V_2$. Suppose this is not true for some $v_2 \in V_2$ and $u_1 \in U_1$. Then we have $v_2u_1 \in A(H)$. Let $v_3 \in V_3$ be arbitrary and $u_2 \in U_2 \cap N^+(v_3)$. Then $H[\{u_1, v_3, u_2, v_2\}]$ is isomorphic to C'_4 or C''_4 . This completes our proof that $UN(H_2)$ is a complete bipartite graph.

Case 3: $d_1 = 3$.

Consider a Min-Max ordering π for H_1 . Let $\pi(x) = \min\{\pi(x') : x' \in V\}$, $\pi(t) = \max\{\pi(t') : t' \in U\}$, $\pi(y) = \max\{\pi(y') : y' \in U, x \rightarrow y'\}$, and $\pi(z) = \min\{\pi(z') : z' \in V, z' \rightarrow t\}$.

Let $T = N^-(t)$. Since H is strong, every vertex of V has an out-neighbor. Since π is a Min-Max ordering, $z' \rightarrow t$ for each $z' \in V$ with $\pi(z') \geq \pi(z)$. Thus, $T = \{z' \in V : \pi(z') \geq \pi(z)\}$. Let $X = N^+(x)$. Since H is strong, every vertex of U has an in-neighbor. Since π is a Min-Max ordering, $x \rightarrow y'$ for each $y' \in U$ with $\pi(y') \leq \pi(y)$. Thus, $X = \{y' \in U : \pi(y') \leq \pi(y)\}$.

Since $\text{dist}(x, z) = 2$ in $UN(H_1)$, we have $z \rightarrow y'$ for some $y' \in N^+(x)$. Since π is a Min-Max ordering, $z \rightarrow y$ (consider the arcs zy' and xy). Now for every $z'' \in V$ with $\pi(z'') \leq \pi(z)$, we have $z'' \rightarrow y$ (as π is a Min-Max ordering). Similarly, for every $y'' \in U$ with $\pi(y'') \geq \pi(y)$, we have $z \rightarrow y''$.

Let $\pi(w) = \min\{\pi(w') : w' \in U\}$. Notice that $\text{dist}(w, t) = 2$ in $UN(H_1)$. Thus, for some $z' \in T$, we have $z' \rightarrow w$. Hence, $z \rightarrow w$ and $z \rightarrow X$ (as π is a Min-Max ordering). We conclude that $z \rightarrow U$. Similarly, we can obtain that $V \rightarrow y$.

Let $Y = V - T$ and $Z = U - X$. Let $x' \in X$, $y' \in Y$. We have $z \rightarrow x'$ and $y' \rightarrow y$. Hence (as π is a Min-Max ordering), $y' \rightarrow x'$. Thus, $Y \rightarrow X$. Analogously, we can prove that $T \rightarrow Z$.

Let $x' \in X$, $t' \in T$, $y' \in Y$ and $x' \rightarrow t'$. Then $t't$, ty' , $y'x' \in A(H)$. Since $H[\{x', t, t', y'\}]$ is not isomorphic to \vec{C}_4 , C'_4 or C''_4 , we have $x' \rightarrow y'$ and $t \rightarrow t'$. Thus, if $x' \rightarrow t'$, we have $x' \rightarrow Y$. Analogously, if $x' \rightarrow t'$, we have $Z \rightarrow t'$. Hence,

$$x' \rightarrow t' \text{ implies } x' \rightarrow Y \text{ and } Z \rightarrow t' \quad (3)$$

We will now prove the following: for a pair u, u' of distinct vertices of U we have $N^+(u) \subseteq N^+(u')$ or $N^+(u') \subseteq N^+(u)$. By Lemma 3.1, this implies that H has a 2-Min-Max ordering and we are done. Suppose that we have neither $N^+(u) \subseteq N^+(u')$ nor $N^+(u') \subseteq N^+(u)$. Thus, there is a pair v, v' of vertices in V such that $u \rightarrow v$, $u' \rightarrow v'$, but uv' and $u'v$ are not arcs in H . Since $H[\{u, u', v, v'\}]$ is not isomorphic to \vec{C}_4 , C'_4 and C''_4 , we have $v \rightarrow u$ and $v' \rightarrow u'$. Now consider four cases.

Case 3.1: $v, v' \in Y$. Let $t' \in T$. By the definition of t , $t \notin \{u, u'\}$. If $u' \rightarrow t'$, then $H[\{t, t', v, u'\}]$ is isomorphic to \vec{C}_4 , C'_4 or C''_4 , which is impossible. Thus, $u't' \notin A(H)$ and $t' \rightarrow u'$. Analogously, $ut' \notin A(H)$ and $t' \rightarrow u$. By the fact that $t' \rightarrow t$ and the existence and nonexistence of previously considered arcs, we conclude that $H[\{v, v', u, u', t, t'\}]$ is

isomorphic to N_1 or N_2 , which is impossible.

Case 3.2: $u, u' \in Z$. We can show that this case is impossible similarly to Case 3.1 but considering x, w instead of t, t' .

Case 3.3: $v \in Y, v' \in T$. By (3), $u' \in Z$. By Case 3.2, we may assume that $u \in X$. Then $H[\{v, v', u', t\}]$ is isomorphic to \vec{C}_4, C'_4 or C''_4 , which is impossible.

Case 3.4: $v, v' \in T$. By Case 3.2, we may assume that $u \in X$. By (3), $Z \rightarrow v$ and, thus, $u' \in X$. By (3), we conclude that $uxu, u'xu', vtv$ and $v'tv'$ are 2-cycles. Notice that $t \rightarrow x$, but $xt \notin A(H)$. Now it follows that $H[\{x, v, v', u, u', t\}]^{\leftrightarrow}$ is isomorphic to C_6 , a contradiction.

It follows from Cases 1,2 and 3 that if neither $UN(H_1)$ nor $UN(H_2)$ are complete bipartite graphs, then we must have $d_1 = d_2 = 3$. In this case we have shown that for every pair u, u' of distinct vertices of U we have $N^+(u) \subseteq N^+(u')$ or $N^+(u') \subseteq N^+(u)$. However, by swapping the roles of U and V we also get that for every pair v, v' of distinct vertices of V we have $N^+(v) \subseteq N^+(v')$ or $N^+(v') \subseteq N^+(v)$. \diamond

The following theorem shows when $\text{MinHOM}(H)$ is polynomial time solvable for the case when H is not strong, and does not contain \vec{C}_4 as an induced subdigraph.

Theorem 3.3 *Let $H = (U, V; A)$ be a semicomplete bipartite digraph with strong components C_1, C_2, \dots, C_p ($p \geq 2$) satisfying the following:*

- *There is no arc from C_i to C_j for $i > j$,*
- *H does not contain an induced subdigraph belonging to \mathcal{HFORB} or an induced directed 4-cycle.*

Then H has a 2-Min-Max ordering and $\text{MinHOM}(H)$ is polynomial time solvable.

Proof: Suppose there are $v, v' \in V$ and $u, u' \in U$ such that $A(H \rightarrow [\{v, v', u, u'\}]) = \{vu, v'u'\}$. Note that $\{v, v', u, u'\}$ belong to a strong component of H as they are contained in a 4-cycle. Let $\{v, v', u, u'\} \subseteq V(C_t)$ for some t and let $V_1 = \{x \in V \mid x \in C_i, i < t\}$, $U_1 = \{y \in U \mid y \in C_i, i < t\}$, $V_2 = V \cap C_t$, $U_2 = U \cap C_t$, $V_3 = V - V_1 - V_2$ and $U_3 = U - U_1 - U_2$.

If there is a $w' \in U_3$ and $w \in V_3$, such that $w' \rightarrow w$, then $H[\{u, v, u', v', w, w'\}]$ is either the dual of N_1 or N_2 . Therefore we must have $A(H[U_3 \cup V_3]) = V_3 \times U_3$. Analogously we must have $A(H[U_1 \cup V_1]) = V_1 \times U_1$. Consider $H' = H[U_2 \cup V_2]$ and note that H' is strong and does not contain a digraph from \mathcal{HFORB} or \vec{C}_4 as an induced subdigraph. Therefore Theorem 3.2 implies that $U_2 \rightarrow V_2$ (as $V_2 \rightarrow U_2$ is not true), which furthermore implies that $(U_1 \cup U_2) \rightarrow (V_2 \cup V_3)$.

Let π be a min-max ordering of H^{\rightarrow} . Let $uv \in A(H)$ and $u'v' \in A(H)$ be two distinct arcs from U to V . As $d^+(u) > 0$ we note that $u \in U_1 \cup U_2$, by the above. Analogously we note that $u' \in U_1 \cup U_2$, $v \in V_2 \cup V_3$ (as $d^-(v) > 0$) and $v' \in V_2 \cup V_3$. By the above we therefore have $uv', u'v \in A(H)$. As u, u', v, v' were chosen arbitrarily, this implies that π is a 2-Min-Max ordering.

If there are no $v, v' \in V$ and $u, u' \in U$ such that $A(H^{\rightarrow}[\{v, v', u, u'\}]) = \{vu, v'u'\}$. Then H^{\rightarrow} satisfies the condition of Lemma 3.1. Therefore H has the 2-Min-Max ordering. \diamond

The following theorem shows when $\text{MinHOM}(H)$ is polynomial time solvable for the case when H does contain \vec{C}_4 as an induced subdigraph.

Theorem 3.4 *Let H be a semicomplete bipartite digraph. Assume that H does not contain a digraph from \mathcal{HFORB} as an induced subdigraph, but contains \vec{C}_4 as an induced subdigraph. Then H is an extension of a \vec{C}_4 and $\text{MinHOM}(H)$ is polynomial time solvable.*

Proof: Observe that an extension L of any cycle \vec{C}_p , $p \geq 2$, has a p -Min-Max ordering. Thus, $\text{MinHOM}(L)$ is polynomial time solvable by Theorem 2.2. Let $C = v_1u_1v_2u_2v_1$ be an induced 4-cycle of H . It suffices to prove that H is an extension of C .

For $i = 1, 2$, let $M^+(v_i) = \{u \in U : v_iu \in A(H), uv_i \notin A(H)\}$, $M^-(v_i) = \{u \in U : uv_i \in A(H), v_iu \notin A(H)\}$, and $M(v_i) = \{u \in U : uv_i, v_iu \in A(H)\}$. We have $M^+(v_1) \cap M^+(v_2) = \emptyset$ as otherwise $H[\{v_1, v_2, u_1, u_2, u_3\}] \cong H^*$, where $u_3 \in N^+(v_1) \cap N^+(v_2)$. We have $M^-(v_1) \cap M^-(v_2) = \emptyset$ as otherwise $H[\{v_1, v_2, u_1, u_2, u_3\}] \cong H^{**}$, where $u_3 \in N^-(v_1) \cap N^-(v_2)$ and H^{**} is the converse of H^* .

We have $M(v_1) \cap (M^+(v_2) \cup M^-(v_2)) \neq \emptyset$ as otherwise $H[\{v_1, u_1, v_2, u\}] \cong C'_4$, where $u \in M(v_1) \cap M^+(v_2)$ or $H[\{v_1, u_2, v_2, u\}] \cong C'_4$, where $u \in M(v_1) \cap M^-(v_2)$. Moreover, $M(v_1) \cap M(v_2) = \emptyset$ as otherwise $H[\{v_1, u_1, v_2, u\}] \cong C''_4$, where $u \in M(v_1) \cap M(v_2)$. The arguments above imply that $M(v_1) = M(v_2) = \emptyset$, and $M^+(v_1) = M^-(v_2)$ and $M^-(v_1) = M^+(v_2)$.

Similarly, we can define $M^+(u_i)$, $M^-(u_i)$ and $M(u_i)$, $i = 1, 2$, and prove the relations analogous to those for $M^+(v_i)$, $M^-(v_i)$ and $M(v_i)$, $i = 1, 2$.

Let $v \in V - \{v_1, v_2\}$ and $u \in U - \{u_1, u_2\}$ be arbitrary. Without loss of generality, assume that $u \in M^+(v_1) = M^-(v_2)$ and $v \in M^+(u_2) = M^-(u_1)$ (all other cases can be treated similarly). To show that H is an extension of C , it suffices to prove that $v \rightarrow u$, but $uv \notin A(H)$. Suppose first that $u \rightarrow v$ and $v \rightarrow u$. Then $H[\{v, u, v_2, u_2\}] \cong C'_4$, a contradiction. Now suppose that $u \rightarrow v$, but $vu \notin A(H)$. Then $H[\{v, v_1, v_2, u, u_2\}] \cong H^*$, a contradiction. Thus, $v \rightarrow u$, but $uv \notin A(H)$ and we are done. \diamond

The three theorems of this section and the fact that \vec{C}_4 has a 4-Min-Max ordering imply the following:

Corollary 3.5 *Let H be a connected semicomplete bipartite digraph not containing a digraph from \mathcal{HFORB} as an induced subdigraph. Then $\text{MinHOM}(H)$ is polynomial time solvable and H has a k -Min-Max ordering for $k = 2$ or 4 .*

4 NP-hardness Cases

It is well known that the problem of finding a maximum size independent set in an undirected graph G is NP-hard. We say that a set I in a digraph D is *independent* if no vertices in I are adjacent. Clearly, the problem of finding a maximum size independent set in a digraph D (MISD) is NP-hard.

Lemma 4.1 *$\text{MinHOM}(C'_4)$ is NP-hard.*

Proof: Let H be isomorphic to C'_4 as follows: $V(H) = \{1, 2, 3, 4\}$ and $A(H) = \{12, 21, 23, 34, 41\}$. Let D be an arbitrary digraph. We replace every arc uv of D by the digraph G_{uv} with $V(G_{uv}) = \{x, y, z, u, v\}$ and $A(G_{uv}) = \{ux, xy, yz, vz\}$. Let D' be the obtained digraph. Define the cost function as follows: $c_1(u) = 1, c_2(u) = 1, c_3(u) = 0, c_4(u) = 1, c_1(v) = 1, c_2(v) = 1, c_3(v) = 0, c_4(v) = 1, c_i(t) = 0$ when $t \notin \{u, v\}$ and $i \in \{1, 2, 3, 4\}$.

Let h be a mapping from $V(G_{uv})$ to $V(H)$, and let uv be an arc in D . If $h(u) = h(y) = h(v) = 1$ and $h(x) = h(z) = 2$, then h is a homomorphism. Thus, there is a homomorphism of D' to H , which maps all vertices of D into 1 and the vertices of D' not in D into 1 or 2.

Now let f be a homomorphism of G_{uv} to H . Observe that if $f(u) = 1$, then $f(x) = 2, f(y)$ is either 1 or 3, $f(z)$ is either 2 or 4, $f(v)$ is either 1 or 3. . Similarly if $f(u) = 3$, then $f(v) = 1$; if $f(u) = 4$, then $f(v)$ is either 2 or 4; if $f(u) = 2$, then $f(v)$ is either 2 or 4.

Let g be a minimum cost homomorphism of D' to H and let $S = \{s \in V(D) : g(s) = 3\}$. Notice that the cost of g is $|V(D)| - |S|$ and S is an independent set by the arguments of the previous paragraph. Thus, S is an independent set of D of maximum size.

Let I be a maximum size independent set in D . The above arguments show that there is a homomorphism d of D' to H such that $d(t) = 3$ if $t \in I$ and $d(t) = 1$ if $t \in V(D) - I$. Notice that d is a minimum cost homomorphism.

Now we can conclude that $\text{MinHOM}(H)$ is NP-hard since MISD is NP-hard. \diamond

Lemma 4.2 *$\text{MinHOM}(C''_4)$ is NP-hard.*

Proof: Let H be isomorphic to C_4'' as follows: $V(H) = \{1, 2, 3, 4\}$, $A(H) = \{12, 21, 23, 32, 34, 41\}$. Let D be an arbitrary digraph. We replace every arc uv of D by the digraph G_{uv} with $V(G_{uv}) = \{x, u, v\}$ and $A(G_{uv}) = \{ux, xv\}$. Let D' be the obtained digraph. Define the cost function as follows: $c_1(u) = 1$, $c_2(u) = 1$, $c_3(u) = 1$, $c_4(u) = 0$, $c_1(v) = 1$, $c_2(v) = 1$, $c_3(v) = 1$, $c_4(v) = 0$, $c_i(x) = 0$ for $i = 1, 2, 3, 4$.

Let h be a mapping from $V(G_{uv})$ to $V(H)$, and let uv be an arc in D . If $h(u) = h(v) = 1$ and $h(x) = 2$, then h is a homomorphism. Thus, there is a homomorphism of D' to H , which maps all vertices of D into 1 and the vertices of D' not in D into 2.

Now let f be a homomorphism of G_{uv} to H . Observe that if $f(u) = 1$, then $f(x) = 2$ and $f(v)$ is either 1 or 3. Similarly if $f(u) = 2$, then $f(v) \in \{2, 4\}$; if $f(u) = 3$, then $f(v) \in \{1, 3\}$; if $f(u) = 4$, then $f(v) = 2$.

Let g be a minimum cost homomorphism of D' to H and let $S = \{s \in V(D) : g(s) = 4\}$. Notice that the cost of g is $|V(D)| - |S|$ and S is an independent set by the arguments of the previous paragraph. Thus, S is an independent set of D of maximum size.

The rest of the proof is similar to that of Lemma 4.1. ◇

The following lemma was stated in [5]. We give a proof here for the sake of completeness.

Lemma 4.3 *MinHOM(H^*) is NP-hard.*

Proof: Let H be isomorphic to H^* as follows: $V(H) = \{1, 2, 3, 4, 5\}$, $A(H) = \{12, 23, 34, 41, 15, 35\}$. We replace every arc uv of D by the digraph G_{uv} with $V(G_{uv}) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where $u = v_6$ and $v = v_7$, and $A(G_{uv}) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_5v_6, v_5v_7, v_1v_6, v_3v_7\}$. Let D' be the obtained digraph. Define the cost function as follows: $c_1(v_6) = 1$, $c_2(v_6) = 1$, $c_3(v_6) = 1$, $c_4(v_6) = 0$, $c_5(v_6) = 1$, $c_1(v_7) = 1$, $c_2(v_7) = 1$, $c_3(v_7) = 1$, $c_4(v_7) = 0$, $c_5(v_7) = 1$, and $c_i(v_j) = 0$ for each $j \neq 6, 7$.

Let h be a mapping from $V(G_{uv})$ to $V(H)$, and let uv be an arc in D . If $h(v_i) = i$ for each $i = 1, 2, 3, 4$, $h(v_5) = 1$ and $h(u) = h(v) = 5$, then h is a homomorphism. Thus, there is a homomorphism of D' to H , which maps all vertices of D into 5.

Now let f be a homomorphism of G_{uv} to H . Observe that if $x = f(u) = f(v)$, then $x = 5$. Also, if $f(u) = 5$ then $f(v) \in \{2, 4, 5\}$ and if $f(v) = 5$ then $f(u) \in \{2, 4, 5\}$.

Let g be a minimum cost homomorphism of D' to H and let $S = \{s \in V(D) : g(s) = 4\}$. Notice that the cost of g is $|V(D)| - |S|$ and S is an independent set by the arguments of the previous paragraph. Thus, S is an independent set of D of maximum size.

The rest of the proof is similar to that of Lemma 4.1. ◇

Lemma 4.4 *MinHOM(N_1) is NP-hard.*

Proof: We shall reduce the maximum independent set problem to $MinHOM(N_1)$. Let H be the following digraph isomorphic to N_1 : $V(H) = \{1, 2, 3, 4, 5, 6\}$,

$$A(H) = \{12, 21, 34, 43, 56, 65, 23, 25, 14, 16, 54, 36\}.$$

Let D be an arbitrary digraph. We replace every arc uv of D with the digraph G_{uv} with $V(G_{uv}) = \{x, y, z, u, v\}$ and $A(G_{uv}) = \{ux, vz, xy, yz\}$. Consider the following cost function: $c_2(u) = c_2(v) = 1$, $c_6(u) = c_6(v) = 0$, $c_i(u) = c_i(v) = 2M + 1$ for $i \neq 2, 6$ and $c_6(y) = 2M + 1$, where $M = |V(D)|$. In all remaining cases the cost is zero. Let D' be the obtained digraph, let f be a mapping from $V(D')$ to $V(H)$, and let uv be an arc in D .

Assume that $f(u) = f(v) = 2$. Then with $f(x) = f(z) = 1$ and $f(y) = 2$, we obtain a homomorphism from G_{uv} to H of cost 2. This implies there is a homomorphism of D' to H of cost $2M < 2M + 1$, and, thus, every vertex of D in D' must be colored either 2 or 6 in any minimum cost homomorphism of D' to H . Let f be a homomorphism of D' to H and let us consider the remaining options for coloring the vertices of D in D' .

Assume that $f(v) = 6$ and $f(u) = 2$. Then with $f(z) = 5$, $f(y) = 2$ and $f(x) = 1$, we obtain a homomorphism from G_{uv} to H of cost 1. Assume that $f(v) = 2$ and $f(u) = 6$. Then with $f(x) = 5$, $f(y) = 4$ and $f(z) = 3$, we obtain a homomorphism from G_{uv} to H of cost 1. Note that if $f(u) = f(v) = 6$, then $f(x) = f(z) = 5$ and $f(y) = 6$. Then the cost of f will be at least $2M + 1$ implying we cannot color both vertices u and v in color 6 in any minimum cost homomorphism of D' to H .

Now let f be a minimum cost homomorphism, let S be the vertices of D in D' colored 6 and $T = V(D) - S$. Recall that the vertices of T are colored 2. Notice that S is an independent set and the cost of f equals $|T|$.

The rest of the proof is similar to that of Lemma 4.1. ◇

Lemma 4.5 *MinHOM(N_2) is NP-hard.*

We shall reduce the maximum independent set problem to $MinHOM(N_2)$. Let H be the following digraph isomorphic to N_2 : $V(H) = \{1, 2, 3, 4, 5, 6\}$,

$$A(H) = \{12, 34, 43, 56, 65, 23, 25, 14, 16, 54, 36\}.$$

Let D be an arbitrary digraph. We replace every arc uv of D by the digraph G_{uv} with $V(G_{uv}) = \{x, y, z, u, v\}$ and $A(G_{uv}) = \{ux, vz, xy, zy\}$. We introduce the following cost function: $c_1(u) = c_1(v) = 1$, $c_5(u) = c_5(v) = 0$, $c_i(u) = c_i(v) = 2M + 1$ for $i \neq 1, 5$ and $c_4(x) = 2M + 1$ and $c_6(z) = 2M + 1$, where $M = |V(D)|$. In any other cases the cost is zero. Let D' be the obtained digraph, let f be a mapping from $V(D')$ to $V(H)$, and let uv be an arc in D .

Assume that $f(u) = f(v) = 1$. With $f(x) = f(z) = 2$ and $f(y) = 3$ with obtain a homomorphism from H' to H with cost 2. Thus, there is a homomorphism of D' to H of

cost at most $2M$ (assign all vertices of D in D' color 2) and no vertex of D in D' must not be assigned any color other than 1 and 5. Let f be a homomorphism of D' to H and let us consider the remaining options for coloring the vertices of D in D' .

Assume that $f(v) = 1$ and $f(u) = 5$. With $f(z) = 2$, $f(x) = 6$ and $f(y) = 5$, we obtain a homomorphism of G_{uv} to H of cost 1. Assume that $f(v) = 5$ and $f(u) = 1$. With $f(x) = 2$, $f(z) = 4$ and $f(y) = 3$, we obtain a homomorphism of G_{uv} to H of cost 1. Note that if $f(u) = f(v) = 5$, then $f(x) \in \{4, 6\}$ and $f(z) \in \{4, 6\}$. Thus, f has cost at least $2M + 1$ implying that a minimum cost homomorphism of D' to H does not assign adjacent vertices of D color 5 (in D').

Now let f be a minimum cost homomorphism, let S be the vertices of D in D' colored 5 and $T = V(D) - S$. Recall that the vertices of T are colored 1. Notice that S is an independent set and the cost of f equals $|T|$.

The rest of the proof is similar to that of Lemma 4.1. ◇

Corollary 4.6 *MinHOM(H) is NP-hard for every $H \in \mathcal{HFORB}$.*

Proof: If H is isomorphic to C'_4 , C''_4 , H^* , N_1 or N_2 or the converse of one of the five digraphs, then MinHOM(H) is NP-hard due to the lemmas of this section and the simple fact that if MinHOM(H) is NP-hard and H' is the converse of H then MinHOM(H') is NP-hard as well.

Let \mathcal{B} be the set consisting of the following bipartite graphs: bipartite claw, bipartite net, bipartite tent and every even cycle with at least 6 vertices. If $UN(H^s)$, where $s \in \{\rightarrow, \leftarrow\}$, is isomorphic to a graph in \mathcal{B} , then MinHOM(H) is NP-hard due to Theorem 1.2 and the transformation from a bipartite undirected graph to a semicomplete bipartite digraph described in the last paragraph of subsection ‘Minimum Cost Homomorphisms’ of Section 1. If $UN(H^{\leftrightarrow})$ is isomorphic to a graph in \mathcal{B} , then MinHOM(H) is NP-hard as, for each bipartite undirected graph L , MinHOM(L) is equivalent to MinHOM(L^+), where L^+ is the digraph obtained from L by replacing every edge xy with two arcs xy and yx . ◇

5 Dichotomy for semicomplete multipartite digraphs

A digraph D is called *semicomplete k -partite* if D can be obtained from a complete k -partite (undirected) graph G by replacing every edge xy of G by either the arc xy or the arc yx or the pair xy, yx of arcs. Let TT_p denote the acyclic tournament on $p \geq 1$ vertices. Let $p \geq 3$ and let TT_p^- be a digraph obtained from TT_p by deleting the arc from the vertex of in-degree zero to the vertex of out-degree zero. Combining the main result of this paper with the main result of [5], we obtain the following:

Theorem 5.1 *Let H be a semicomplete k -partite digraph. If $k = 2$ and H does not*

contain a digraph from \mathcal{HFORB} as an induced subdigraph or if $k \geq 3$ and H is an extension of either TT_k or TT_{k+1}^- or \vec{C}_3 , then $\text{MinHOM}(H)$ is polynomial time solvable. Otherwise, $\text{MinHOM}(H)$ is NP-hard.

6 Further Research

In the case of undirected graphs H , the well-known theorem of Hell and Nešetřil [9] on the homomorphism problem implies that $\text{MinHOM}(H)$ is NP-hard for each non-bipartite graph H . The authors of [3] obtained a complete dichotomy of the computational complexity of $\text{MinHOM}(H)$ when H is undirected. The dichotomy obtained in this paper significantly extends the dichotomy of [3]. This indicates that the problem of obtaining a dichotomy for the computational complexity of $\text{MinHOM}(H)$ when H is a bipartite digraph is a very difficult problem. Note that $\text{MinHOM}(H)$ is polynomial-time solvable for some non-bipartite digraphs, for example, for acyclic tournaments [5]. Thus, a dichotomy for bipartite directed case does not coincide with a dichotomy for the general directed case. The problem of obtaining dichotomy for both cases is a very interesting open problem.

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