A primal-dual approximation algorithm for the *k*-prize-collecting minimum vertex cover problem with submodular penalties

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Problems & Ideas

- the k-prize-collecting minimum vertex cover problem with submodular penalties:

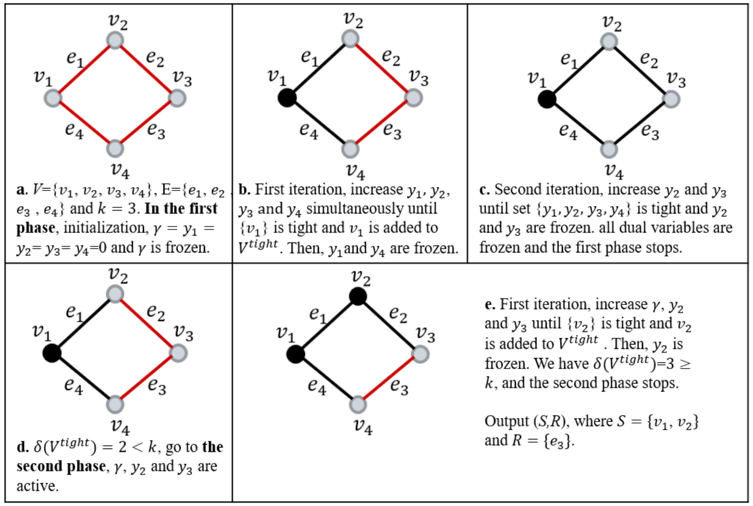
  Given a graph G=(V,E), an integer k, a cost function c and a submodular penalty function π. This problem finds a (S,R) such that at least k edges are covered by S and min(c(S)+π(R)).

- Ideas: 1. guess the maximum cost vertex v_{max}^* in an optimal solution; 2. construct a feasible solution based on the two-phase primal-dual algorithm and v_{max}^*.

Algorithm 1: The two-phase primal-dual algorithm

Input: An instance $\mathcal{I} = (G; c, \pi; k)$ of the k-PCVCS.
Output: A feasible pair (S, R).

Set $V^{tight} = \emptyset$, $E^{act} = E$, $y_e = 0$ for any $e \in E$ and $\gamma = 0$.
while $E^{act} \neq \emptyset$ **do**
  Keep $\gamma = 0$ and increase $\{y_e\}_{e \in E^{act}}$ simultaneously until either some vertex v becomes tight or some edge set E' becomes tight.
  if *vertex v become tight* **then**
    $V^{tight} := V^{tight} \cup \{v\}$, $E^{act} := E^{act} \setminus \delta(\{v\})$.
  else
    $E^{act} := E^{act} \setminus E'$.
while $|\delta(V^{tight})| < k$ **do**
  Increase $\{y_e\}_{e \in E \setminus \delta(V^{tight})}$ and γ simultaneously until some vertex v becomes tight.
  $V^{tight} := V^{tight} \cup \{v\}$.
$S := V^{tight}$ and $R := E \setminus \delta(V^{tight})$. Output (S, R).



Left: the two-phase primal-dual algorithm.; Right: **a**, **b** and **c** are the Phase 1 of the two-phase primal-dual algorithm, and **e** and **f** are the Phase 2 of the two-phase primal-dual algorithm

Main Contributions

- Contributions:
  - We proposed the k-prize-collecting minimum vertex cover problem with submodular penalties, which is a generalization of many well-known vertex cover problem;
  - We presented a two-phase algorithm based on the guessing technique and the primal-dual framework;
  - We proved that the approximation factor of this algorithm is 3, when the submodular penalty cost function is normalized and nondecreasing. In particular, when the submodular penalty cost function is linear, we proved that its approximation factor is 2.

Algorithm 2:

Input: An instance $\mathcal{I} = (G; c, \pi; k)$ of the k-PCVCS.
Output: A feasible pair (S, R).

1 **for** $v \in V$ **do**
2   Construct the auxiliary instance $\mathcal{I} \setminus \{v\}$;
3   Using Algorithm 1, find a feasible solution (S_v, R_v) of instance $\mathcal{I} \setminus \{v\}$
4 Let $v' = \arg\min_{v \in V}(c(S_v) + \pi(S_v) + c(v))$, and output $(S, R) = (S_{v'} \cup \{v'\}, R_{v'})$.

The two-phase algorithm for the *k*-prize-collecting minimum vertex cover problem with submodular penalties.