Supplementary Material for "Estimation of component reliability from superposed renewal processes by means of latent variables"

Rodrigues, Agatha S. (Department of Statistics, Federal University of Espirito Santo, Vitoria, ES, Brazil) - agatha.rodrigues@ufes.br
Kerschke, Pascal (Information Systems and Statistics, University of Münster, 48149 Münster, Germany)
Pereira, Carlos A. de B. (Institute of Mathematics and Statistics, University of São Paulo, São Paulo, SP, Brazil)
Trautmann, Heike (Information Systems and Statistics, University of Münster, 48149 Münster, Germany)
Wagner, Carolin (Information Systems and Supply Chain Management, University of Münster, 48149 Münster, Germany
Hellingrath, Bernd (Information Systems and Supply Chain Management, University of Münster, 48149 Münster, Germany
Polpo, Adriano (Western Australia University, Perth, Australia.)

Journal: Computational Statistics

This supplementary material was built to assist the content of the article "Estimation of component reliability from superposed renewal processes by means of latent variables". Section 1 presents the distribution of the latent indicator vector ($\boldsymbol{d}$) conditional to the observed data ($\boldsymbol{\mathcal{T}}$) in a case that there are five failures ($r_i = 5$). Section 2 presents the study of the number of Monte Carlo samples in the EM-algorithm.

1 Conditional distribution of d for $r_i = 5$

For a fixed i, $f(\boldsymbol{d}_i \mid \boldsymbol{\mathcal{T}}_i)$ can be written as

$$\begin{aligned} & f(\boldsymbol{d}_i \mid \boldsymbol{\mathcal{T}}_i) = f(d_{1i}, d_{2i}, \ldots, d_{r_i i} \mid \boldsymbol{\mathcal{T}}_i) \\ = \; & f(d_{r_i i} \mid \boldsymbol{\mathcal{T}}_i, d_{(r_i-1)i}, d_{(r_i-2)i}, \ldots, d_{2i}, d_{1i}) f(d_{(r_i-1)i} \mid \boldsymbol{\mathcal{T}}_i, d_{(r_i-2)i}, \ldots, d_{2i}, d_{1i}) \\ & \times \ldots \times f(d_{2i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i}) f(d_{1i} \mid \boldsymbol{\mathcal{T}}_i). \end{aligned} \tag{1}$$

The conditional distribution of d given $\mathcal{T}$ is presented in (1) for any number of failures $r_i \geq 1$. In order to present the idea, consider a particular case with $r_i = 5$ as an example. Thus, $\boldsymbol{\mathcal{T}}_i = (t_{1i}, t_{2i}, t_{3i}, t_{4i}, t_{5i}, \tau_i)$ and

$$\begin{aligned} f(\boldsymbol{d}_i \mid \boldsymbol{\mathcal{T}}_i) &= f(d_{1i}, d_{2i}, d_{3i}, d_{4i}, d_{5i} \mid \boldsymbol{\mathcal{T}}_i) \\ &= f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{4i}, d_{3i}, d_{2i}, d_{1i}) f(d_{4i} \mid \boldsymbol{\mathcal{T}}_i, d_{3i}, d_{2i}, d_{1i}) \\ & \quad f(d_{3i} \mid \boldsymbol{\mathcal{T}}_i, d_{2i}, d_{1i}) f(d_{2i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i}) f(d_{1i} \mid \boldsymbol{\mathcal{T}}_i). \end{aligned} \tag{2}$$

One comment is important to point: when we develop the distribution of d_{vi} conditional to the distribution of the indicator of the socket that each previous failure occurred, that is, conditional to $(d_{(v-1)i}, d_{(v-1)i}, \ldots, d_{2i}, d_{1i})$, with $v = 2, \ldots, m$, one only needs to worry about the number of sockets that the previous failures occurred and the index of the last failure that occurred in each socket that had failure, regardless the socket index, because we are assuming the lifetime distributions of the components at the sockets are i.i.d.

The development of each term of the right side of the Equation (2) is presented in the following.

1.1 The conditional density function $f(d_{1i} \mid \boldsymbol{\mathcal{T}}_i)$

Under i.i.d assumption, the distribution of $d_{1i} = j \mid \boldsymbol{\mathcal{T}}_i$ follows a Multinomial distribution, $Multin(1, \boldsymbol{p}_{1i})$, with $\boldsymbol{p}_{1i} = (p_{11i}, \ldots, p_{1mi})$ and $p_{1ji} = 1/m$, $j = 1, \ldots, m$. Note that in this case, the multinomial distribution equals a discrete uniform distribution.

1.2 The conditional density function $f(d_{2i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i})$

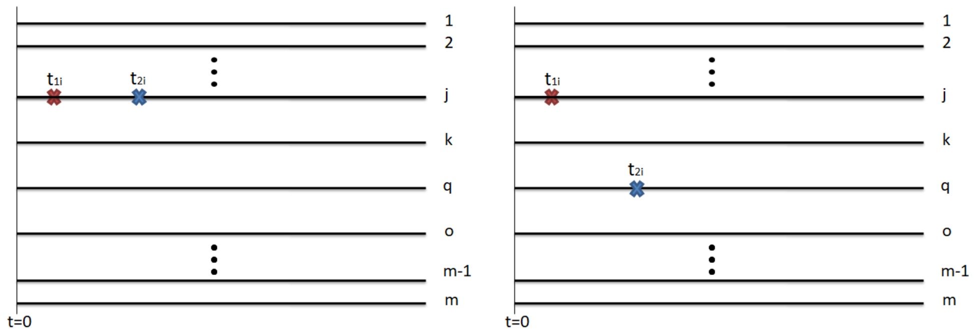
The distribution of d_{2i} conditional to the socket that first failure occurred, say at the socket j (any $j \in \{1, \ldots, m\}$), can be described as follows:

$$f(d_{2i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j) \propto [f(t_{2i} - t_{1i})]^{\mathrm{I(d_{2i}=j)}} \prod_{l=1; l \neq j}^{m} [f(t_{2i})]^{\mathrm{I(d_{2i}=l)}},$$

that is, $d_{2i} \mid (\boldsymbol{t}_i, d_{1i} = j)$ follows $Multin(1, \boldsymbol{p}_{2i})$, in which $\boldsymbol{p}_{2i} = (p_{21i}, \ldots, p_{2mi})$, $p_{2ji} = f(t_{2i} - t_{1i})/C$ and $p_{2li} = f(t_{2i})/C$, $l = 1, \ldots, m$ and $l \neq j$, with $C = f(t_{2i} - t_{1i}) + (m - 1) f(t_{2i})$.

1.3 The conditional density function $f(d_{3i} \mid \boldsymbol{\mathcal{T}}_i, d_{2i}, d_{1i})$

Now we need to specify the distribution of d_{3i} conditional to the values assumed by d_{2i} and d_{1i}. For that, one has to consider the following two situations (Figure 1): 1) the first failure occurs at a given socket, say at the socket j, and the second failure also occurs at the socket j (Figure 1a); and 2) the first failure occurs at a given socket, say at the socket j, and the second failure occurs in a different socket, say at the socket q (Figure 1b), with $j \neq q$, $j = 1, \ldots, m$ and $q = 1, \ldots, m$.



(a) Situation 1 (b) Situation 2

Figure 1: The two possible situations of how two failures can occur in the sockets.

So, depending on each situation, the conditional distribution of d_{3i} is given by:

- Situation 1: distribution of $d_{3i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j)$:

$$f(d_{3i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j) \propto [f(t_{3i} - t_{2i})]^{\mathrm{I}(\mathrm{d_{3i}=j})} \prod_{l=1; l \neq j}^{m} [f(t_{3i})]^{\mathrm{I}(\mathrm{d_{3i}=l})},$$

  that is, $d_{3i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j)$ follows $Multin(1, \boldsymbol{p}_{3i})$, in which $\boldsymbol{p}_{3i} = (p_{31i}, \ldots, p_{3mi})$, $p_{3ji} = f(t_{3i} - t_{2i})/C$ and $p_{3li} = f(t_{3i})/C$, $l = 1, \ldots, m$ and $l \neq j$, with $C = f(t_{3i} - t_{2i}) + (m-1) f(t_{3i})$.

- Situation 2: distribution of $d_{3i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q)$, with $q \neq j$:

$$f(d_{3i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q) \propto [f(t_{3i} - t_{1i})]^{\mathrm{I}(\mathrm{d_{3i}=j})} [f(t_{3i} - t_{2i})]^{\mathrm{I}(\mathrm{d_{3i}=q})} \times \prod_{l=1; l \neq j,q}^{m} [f(t_{3i})]^{\mathrm{I}(\mathrm{d_{3i}=l})},$$

that is, $d_{3i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q)$ follows $Multin(1, \boldsymbol{p}_{3i})$, in which $\boldsymbol{p}_{3i} = (p_{31i}, \ldots, p_{3mi})$, $p_{3ji} = f(t_{3i} - t_{1i})/C$, $p_{3qi} = f(t_{3i} - t_{2i})/C$ and $p_{3li} = f(t_{3i})/C$, $l = 1, \ldots, m$ and $l \neq j, q$, with $C = f(t_{3i} - t_{1i}) + f(t_{3i} - t_{2i}) + (m-2)f(t_{3i})$.

1.4 The conditional density function $f(d_{4i} \mid \boldsymbol{\mathcal{T}}_i, d_{3i}, d_{2i}, d_{1i})$

To specify the distribution of d_{4i} conditional to the values assumed by d_{3i}, d_{2i} and d_{1i}, one has to consider the five situations of how three failures can occur up to three sockets (regardless of which sockets they are), illustrated at Figure 2. They are: 1) all three failures occur at a same socket, say at the socket j (Figure 2a); 2) the first and the second failures occur at a same socket, say at the socket j, and the third failure occurs in a different socket, say at the socket q, with $j \neq q$ (Figure 2b); 3) the first failure occurs at a given socket, say at the socket j, the second failure occurs in a different socket, say at the socket q, and the third failure occurs at the same socket that occurred the first failure (Figure 2c); 4) the first failure occurs at a given socket, say at the socket j, and the second failure occurs in a different socket, say at the socket q, and the third failure occurs at the same socket that occurred the second failure (Figure 2d); 5) each failure occurs in three different sockets (Figure 2e), say that the first failure occurs at socket j, the second occurs at socket q and third failure occurs at socket k, with $j \neq q \neq k$, $j = 1, \ldots, m$, $q = 1, \ldots, m$ and $k = 1, \ldots, m$.

So, depending on each situation, the conditional distribution of d_{4i} is given by:

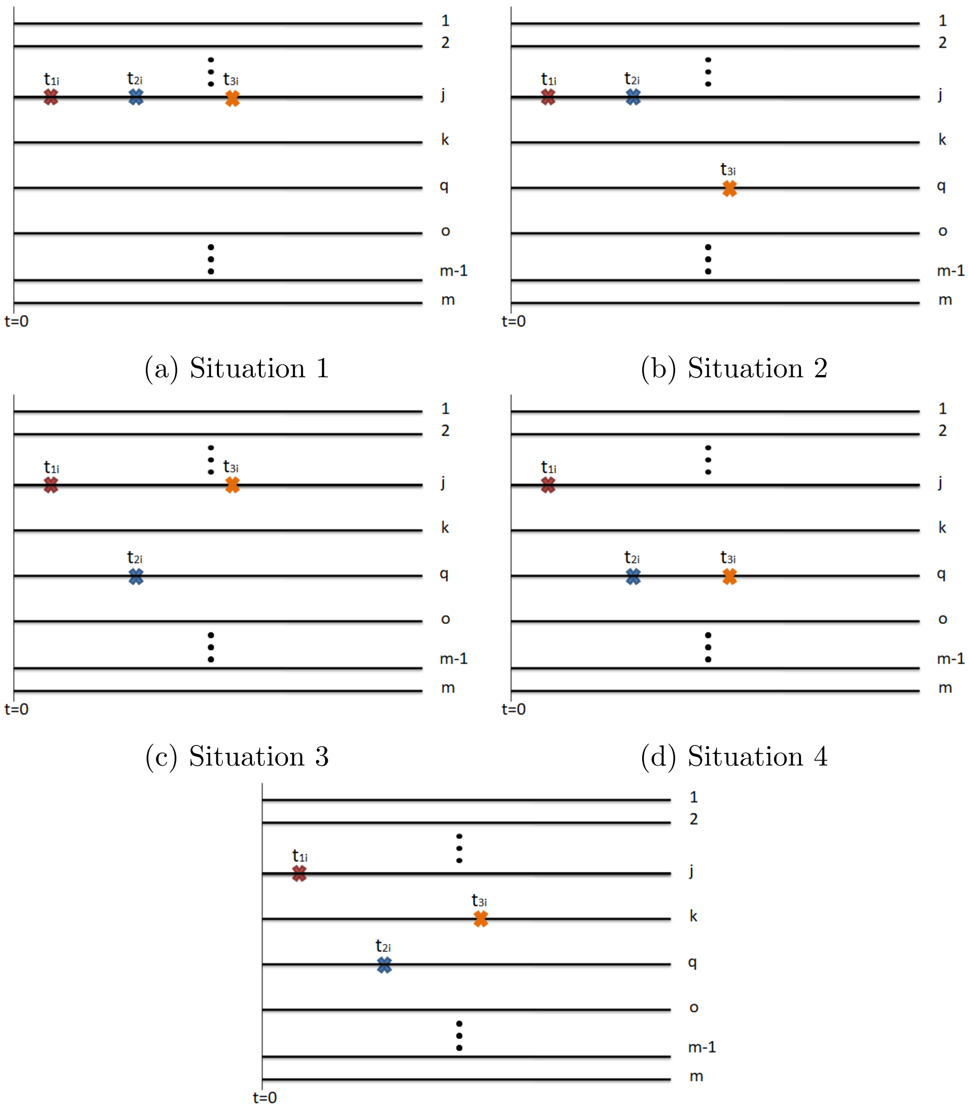
- Situation 1: distribution of $d_{4i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = j)$:

$$f(d_{4i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = j) \propto [f(t_{4i} - t_{3i})]^{\mathrm{I(d_{4i}=j)}} \prod_{l=1;l\neq j}^{m} [f(t_{4i})]^{\mathrm{I(d_{4i}=l)}},$$

  that is, $d_{4i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = j)$ follows $Multin(1, \boldsymbol{p}_{4i})$, in which $\boldsymbol{p}_{4i} = (p_{41i}, \ldots, p_{4mi})$, $p_{4ji} = f(t_{4i} - t_{3i})/C$ and $p_{4li} = f(t_{4i})/C$, $l = 1, \ldots, m$ and $l \neq j$, with $C = f(t_{4i} - t_{3i}) + (m-1)f(t_{4i})$.

- Situation 2: distribution of $d_{4i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = q)$, with $q \neq j$:

$$\begin{aligned} f(d_{4i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = q) \propto \quad & [f(t_{4i} - t_{2i})]^{\mathrm{I(d_{4i}=j)}} [f(t_{4i} - t_{3i})]^{\mathrm{I(d_{4i}=q)}} \\ & \times \prod_{l=1;l\neq j,q}^{m} [f(t_{4i})]^{\mathrm{I(d_{4i}=l)}}, \end{aligned}$$



(a) Situation 1 (b) Situation 2

(c) Situation 3 (d) Situation 4

(e) Situation 5

Figure 2: The five possible situations of how three failures can occur in the sockets.

that is, $d_{4i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = q)$ follows $Multin(1, \boldsymbol{p}_{4i})$, in which $\boldsymbol{p}_{4i} = (p_{41i}, \ldots, p_{4mi})$, $p_{4ji} = f(t_{4i} - t_{2i})/C$, $p_{4qi} = f(t_{4i} - t_{3i})/C$ and $p_{4li} = f(t_{4i})/C$, $l = 1, \ldots, m$ and $l \neq j, q$, with $C = f(t_{4i} - t_{2i}) + f(t_{4i} - t_{3i}) + (m-2)f(t_{4i})$.

- Situation 3: distribution of $d_{4i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = j)$, with

$q \neq j$:

$$f(d_{4i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = j) \propto \quad [f(t_{4i} - t_{3i})]^{\mathrm{I(d_{4i}=j)}}[f(t_{4i} - t_{2i})]^{\mathrm{I(d_{4i}=q)}} \times \prod_{l=1;l\neq j,q}^{m} [f(t_{4i})]^{\mathrm{I(d_{4i}=l)}},$$

that is, $d_{4i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = j)$ follows $Multin(1, \boldsymbol{p}_{4i})$, in which $\boldsymbol{p}_{4i} = (p_{41i}, \ldots, p_{4mi})$, $p_{4ji} = f(t_{4i} - t_{3i})/C$, $p_{4qi} = f(t_{4i} - t_{2i})/C$ and $p_{4li} = f(t_{4i})/C$, $l = 1, \ldots, m$ and $l \neq j, q$, with $C = f(t_{4i} - t_{3i}) + f(t_{4i} - t_{2i}) + (m - 2)f(t_{4i})$.

- Situation 4: distribution of $d_{4i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q)$, with $q \neq j$:

$$f(d_{4i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q) \propto \quad [f(t_{4i} - t_{1i})]^{\mathrm{I(d_{4i}=j)}}[f(t_{4i} - t_{3i})]^{\mathrm{I(d_{4i}=q)}} \times \prod_{l=1;l\neq j,q}^{m} [f(t_{4i})]^{\mathrm{I(d_{4i}=l)}},$$

that is, $d_{4i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q)$ follows $Multin(1, \boldsymbol{p}_{4i})$, in which $\boldsymbol{p}_{4i} = (p_{41i}, \ldots, p_{4mi})$, $p_{4ji} = f(t_{4i} - t_{1i})/C$, $p_{4qi} = f(t_{4i} - t_{3i})/C$ and $p_{4li} = f(t_{4i})/C$, $l = 1, \ldots, m$ and $l \neq j, q$, with $C = f(t_{4i} - t_{1i}) + f(t_{4i} - t_{3i}) + (m - 2)f(t_{4i})$.

- Situation 5: distribution of $d_{4i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k)$, with $q \neq j \neq k$:

$$f(d_{4i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q) \propto [f(t_{4i} - t_{1i})]^{\mathrm{I(d_{4i}=j)}}[f(t_{4i} - t_{2i})]^{\mathrm{I(d_{4i}=q)}} \times [f(t_{4i} - t_{3i})]^{\mathrm{I(d_{4i}=k)}} \prod_{l=1;l\neq j,q,k}^{m} [f(t_{4i})]^{\mathrm{I(d_{4i}=l)}},$$

that is, $d_{4i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k)$ follows $Multin(1, \boldsymbol{p}_{4i})$, in which $\boldsymbol{p}_{4i} = (p_{41i}, \ldots, p_{4mi})$, $p_{4ji} = f(t_{4i} - t_{1i})/C$, $p_{4qi} = f(t_{4i} - t_{2i})/C$, $p_{4ki} = f(t_{4i} - t_{3i})/C$ and $p_{4li} = f(t_{4i})/C$, $l = 1, \ldots, m$ and $l \neq j, q, k$, with $C = f(t_{4i} - t_{1i}) + f(t_{4i} - t_{2i}) + f(t_{4i} - t_{3i}) + (m - 3)f(t_{4i})$.

Since the socket index does not matter (we have the i.i.d. assumption), the situations 2 and 3 present the same contribution for the conditional distribution of $f(d_{4i} \mid \boldsymbol{\mathcal{T}}_i, d_{3i}, d_{2i}, d_{1i})$.

1.5 The conditional density function $f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{4i}, d_{3i}, d_{2i}, d_{1i})$

To specify the distribution of d_{5i} conditional to the values assumed by d_{4i}, d_{3i}, d_{2i} and d_{1i}, one has to consider the fifteen situations of how four failures can occur up to four sockets (regardless of which sockets they are), illustrated at Figures 3 and 4. They are: 1) all four failures occur at the same socket, say at the socket j (Figure 3a); 2) the first failure occurs at a given socket, say at the socket j, and the second failure occurs in a different socket, say at the socket q, and the third and fourth failures occur at the same socket that occurred the first failure (Figure 3b); 3) the first and the second failures occur at a same socket, say at the socket j, and the third failure occurs in a different socket, say at the socket q, and fourth failure occurs at the same socket that occurred the first and second failures (Figure 3c); 4) the first, the second and the third failures occur at a same socket, say at the socket j, and the fourth failure occurs in a different socket, say at the socket q (Figure 3d); 5) the first failure occurs at a given socket, say at the socket j, and the remaining failures occur in a different socket, say at the socket q (Figure 3e); 6) the first and the second failures occur at the same socket, say at the socket j, and the third and the fourth failures occur in a different socket, say at the socket q (Figure 3f); 7) the first failure occurs at a given socket, say socket j, and the second failure occurs in a different socket, say at the socket q, and the third failure occurs at the same socket that occurred the first failure and the fourth failure occurs at the same socket that occurred the second failure (Figure 3g); 8) the first failure occurs at a given socket, say socket j, and the second and the third failures occur in a different socket, say at the socket q, and the fourth failure occurs at the same socket that occurred the first failure (Figure 3h); 9) the first and the second failures occur at the same socket, say at the socket j, and the third failure occurs in a different socket, say at the socket q, and the fourth failure occurs in a socket that there is no previous failure, say at the socket k (Figure 4a); 10) the first failure occurs at a given socket, say socket j, and the second failure occurs in a different socket, say at the socket q, and the third failure occurs at the same socket that occurred the first failure and the fourth failure occurs in a socket that there is no previous failure, say at the socket k (Figure 4b); 11) the first failure occurs at a given socket, say socket j, and the second failure occurs in a different socket, say at the socket q, and the third failure occurs in a socket that there is no previous failure, say at the socket k, and the fourth failure occurs at the same socket that occurred the first failure (Figure 4c); 12) the first failure occurs at a given socket, say at the socket j, and the second and the third failures occur in a different socket, say at the socket q, and the fourth failure occurs in a socket that there is no previous failure, say at the socket k (Figure 4d); 13) the first failure occurs at a given socket, say socket j, and the second failure occurs in a different socket, say at the socket q, and the third failure occurs in a socket that there is no previous failure, say at the socket k, and the fourth failure occurs at the same socket that occurred the second failure (Figure 4e); 14) the first failure occurs at a given socket, say at the socket j, and the second failure occurs in a different socket, say at the socket q, and the third failure and fourth failures occur in a socket that there is no previous failure, say at the socket k (Figure 4f); 15) each failure occurs in four different sockets (Figure 4g), say that the first failure occurs at socket j, the second failure occurs at the socket q, the third failure occurs at socket k and the fourth failure occurs at the socket o, with $j \neq q \neq k \neq o$, $j = 1, \ldots, m$, $q = 1, \ldots, m$, $k = 1, \ldots, m$ and $o = 1, \ldots, m$.

So, depending on each situation, the conditional distribution of d_{5i} is given by:

- Situation 1: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = j, d_{4i} = j)$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = j, d_{4i} = j) \propto \quad [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=j)}} \times \prod_{l=1; l \neq j}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

  that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = j, d_{4i} = j)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{4i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j$, with $C = f(t_{5i} - t_{4i}) + (m-1)f(t_{5i})$.

- Situation 2: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = j, d_{4i} = j)$, with $q \neq j$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = j, d_{4i} = j) \propto [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=j)}} \times [f(t_{5i} - t_{2i})]^{\mathrm{I(d_{5i}=q)}} \prod_{l=1; l \neq j,q}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

  that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = j, d_{4i} = j)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{4i})/C$, $p_{5qi} = f(t_{5i} - t_{2i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q$, with $C = f(t_{5i} - t_{4i}) + f(t_{5i} - t_{2i}) + (m-2)f(t_{5i})$.

- Situation 3: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = q, d_{4i} = j)$, with $q \neq j$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = q, d_{4i} = j) \propto [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=j)}} \times [f(t_{5i} - t_{3i})]^{\mathrm{I(d_{5i}=q)}} \prod_{l=1; l \neq j,q}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = q, d_{4i} = j)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{4i})/C$, $p_{5qi} = f(t_{5i} - t_{3i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q$, with $C = f(t_{5i} - t_{4i}) + f(t_{5i} - t_{3i}) + (m-2)f(t_{5i})$.

- Situation 4: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = j, d_{4i} = q)$, with $q \neq j$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = j, d_{4i} = q) \propto [f(t_{5i} - t_{3i})]^{\mathrm{I(d_{5i}=j)}} \times [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=q)}} \prod_{l=1; l \neq j,q}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = j, d_{4i} = q)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{3i})/C$, $p_{5qi} = f(t_{5i} - t_{4i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q$, with $C = f(t_{5i} - t_{3i}) + f(t_{5i} - t_{4i}) + (m-2)f(t_{5i})$.

- Situation 5: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q, d_{4i} = q)$, with $q \neq j$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q, d_{4i} = q) \propto [f(t_{5i} - t_{1i})]^{\mathrm{I(d_{5i}=j)}} \times [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=q)}} \prod_{l=1; l \neq j,q}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q, d_{4i} = q)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{1i})/C$, $p_{5qi} = f(t_{5i} - t_{4i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q$, with $C = f(t_{5i} - t_{1i}) + f(t_{5i} - t_{4i}) + (m-2)f(t_{5i})$.

- Situation 6: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = q, d_{4i} = q)$, with $q \neq j$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = q, d_{4i} = q) \propto [f(t_{5i} - t_{2i})]^{\mathrm{I(d_{5i}=j)}} \times [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=q)}} \prod_{l=1; l \neq j,q}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = q, d_{4i} = q)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{2i})/C$, $p_{5qi} = f(t_{5i} - t_{4i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q$, with $C = f(t_{5i} - t_{2i}) + f(t_{5i} - t_{4i}) + (m-2)f(t_{5i})$.

- Situation 7: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = j, d_{4i} = q)$, with $q \neq j$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = j, d_{4i} = q) \propto [f(t_{5i} - t_{3i})]^{\mathrm{I(d_{5i}=j)}} \times [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=q)}} \prod_{l=1; l \neq j,q}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = j, d_{4i} = q)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{3i})/C$, $p_{5qi} = f(t_{5i} - t_{4i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q$, with $C = f(t_{5i} - t_{3i}) + f(t_{5i} - t_{4i}) + (m-2)f(t_{5i})$.

- Situation 8: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q, d_{4i} = j)$, with $q \neq j$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q, d_{4i} = j) \propto [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=j)}} \times [f(t_{5i} - t_{3i})]^{\mathrm{I(d_{5i}=q)}} \prod_{l=1; l \neq j,q}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q, d_{4i} = j)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{4i})/C$, $p_{5qi} = f(t_{5i} - t_{3i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q$, with $C = f(t_{5i} - t_{4i}) + f(t_{5i} - t_{3i}) + (m-2)f(t_{5i})$.

- Situation 9: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = q, d_{4i} = k)$, with $q \neq j \neq k$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = q, d_{4i} = k) \propto [f(t_{5i} - t_{2i})]^{\mathrm{I(d_{5i}=j)}} \times [f(t_{5i} - t_{3i})]^{\mathrm{I(d_{5i}=q)}} [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=k)}} \prod_{l=1; l \neq j,q,k}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = j, d_{3i} = q, d_{4i} = k)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{2i})/C$, $p_{5qi} = f(t_{5i} - t_{3i})/C$, $p_{5ki} = f(t_{5i} - t_{4i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q, k$, with $C = f(t_{5i} - t_{2i}) + f(t_{5i} - t_{3i}) + f(t_{5i} - t_{4i}) + (m-3)f(t_{5i})$.

- Situation 10: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = j, d_{4i} = k)$, with $q \neq j \neq k$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = j, d_{4i} = k) \propto [f(t_{5i} - t_{3i})]^{\mathrm{I(d_{5i}=j)}} \times [f(t_{5i} - t_{2i})]^{\mathrm{I(d_{5i}=q)}} [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=k)}} \prod_{l=1; l \neq j,q,k}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = j, d_{4i} = k)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{3i})/C$, $p_{5qi} = f(t_{5i} - t_{2i})/C$, $p_{5ki} = f(t_{5i} - t_{4i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q, k$, with $C = f(t_{5i} - t_{3i}) + f(t_{5i} - t_{2i}) + f(t_{5i} - t_{4i}) + (m-3)f(t_{5i})$.

- Situation 11: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k, d_{4i} = j)$, with $q \neq j \neq k$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k, d_{4i} = j) \propto [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=j)}} \times [f(t_{5i} - t_{2i})]^{\mathrm{I(d_{5i}=q)}} [f(t_{5i} - t_{3i})]^{\mathrm{I(d_{5i}=k)}} \prod_{l=1; l \neq j,q,k}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k, d_{4i} = j)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{4i})/C$, $p_{5qi} = f(t_{5i} - t_{2i})/C$, $p_{5ki} = f(t_{5i} - t_{3i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q, k$, with $C = f(t_{5i} - t_{4i}) + f(t_{5i} - t_{2i}) + f(t_{5i} - t_{3i}) + (m-3)f(t_{5i})$.

- Situation 12: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q, d_{4i} = k)$, with $q \neq j \neq k$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q, d_{4i} = k) \propto [f(t_{5i} - t_{1i})]^{\mathrm{I(d_{5i}=j)}} \times [f(t_{5i} - t_{3i})]^{\mathrm{I(d_{5i}=q)}} [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=k)}} \prod_{l=1; l \neq j,q,k}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = q, d_{4i} = k)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{1i})/C$, $p_{5qi} = f(t_{5i} - t_{3i})/C$, $p_{5ki} = f(t_{5i} - t_{4i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q, k$, with $C = f(t_{5i} - t_{1i}) + f(t_{5i} - t_{3i}) + f(t_{5i} - t_{4i}) + (m-3)f(t_{5i})$.

- Situation 13: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k, d_{4i} = q)$, with $q \neq j \neq k$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k, d_{4i} = q) \propto [f(t_{5i} - t_{1i})]^{\mathrm{I(d_{5i}=j)}} \times [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=q)}} [f(t_{5i} - t_{3i})]^{\mathrm{I(d_{5i}=k)}} \prod_{l=1; l \neq j,q,k}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k, d_{4i} = q)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{1i})/C$, $p_{5qi} = f(t_{5i} - t_{4i})/C$, $p_{5ki} = f(t_{5i} - t_{3i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q, k$, with $C = f(t_{5i} - t_{1i}) + f(t_{5i} - t_{4i}) + f(t_{5i} - t_{3i}) + (m-3)f(t_{5i})$.

- Situation 14: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k, d_{4i} = k)$, with $q \neq j \neq k$:

$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k, d_{4i} = k) \propto [f(t_{5i} - t_{1i})]^{\mathrm{I(d_{5i}=j)}} \\ \times [f(t_{5i} - t_{2i})]^{\mathrm{I(d_{5i}=q)}} [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=k)}} \prod_{l=1; l \neq j,q,k}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

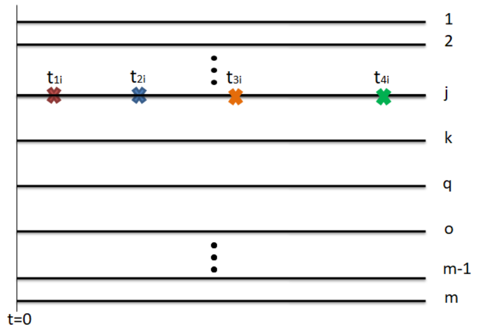
  that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k, d_{4i} = k)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{1i})/C$, $p_{5qi} = f(t_{5i} - t_{2i})/C$, $p_{5ki} = f(t_{5i} - t_{4i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q, k$, with $C = f(t_{5i} - t_{1i}) + f(t_{5i} - t_{2i}) + f(t_{5i} - t_{4i}) + (m-3) f(t_{5i})$.

- Situation 15: distribution of $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k, d_{4i} = o)$, with $q \neq j \neq k \neq o$:

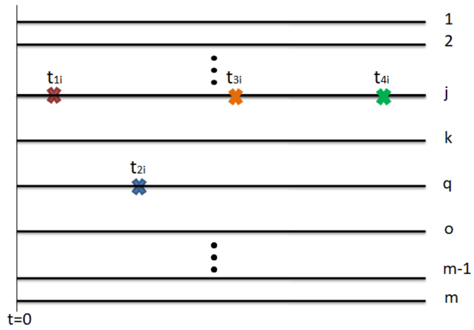
$$f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k, d_{4i} = o) \propto [f(t_{5i} - t_{1i})]^{\mathrm{I(d_{5i}=j)}} \\ \times [f(t_{5i} - t_{2i})]^{\mathrm{I(d_{5i}=q)}} [f(t_{5i} - t_{3i})]^{\mathrm{I(d_{5i}=k)}} [f(t_{5i} - t_{4i})]^{\mathrm{I(d_{5i}=o)}} \\ \times \prod_{l=1; l \neq j,q,k,o}^{m} [f(t_{5i})]^{\mathrm{I(d_{5i}=l)}},$$

  that is, $d_{5i} \mid (\boldsymbol{\mathcal{T}}_i, d_{1i} = j, d_{2i} = q, d_{3i} = k, d_{4i} = o)$ follows $Multin(1, \boldsymbol{p}_{5i})$, in which $\boldsymbol{p}_{5i} = (p_{51i}, \ldots, p_{5mi})$, $p_{5ji} = f(t_{5i} - t_{1i})/C$, $p_{5qi} = f(t_{5i} - t_{2i})/C$, $p_{5ki} = f(t_{5i} - t_{3i})/C$, $p_{5oi} = f(t_{5i} - t_{4i})/C$ and $p_{5li} = f(t_{5i})/C$, $l = 1, \ldots, m$ and $l \neq j, q, k, o$, with $C = f(t_{5i} - t_{1i}) + f(t_{5i} - t_{2i}) + f(t_{5i} - t_{3i}) + f(t_{5i} - t_{4i}) + (m-4) f(t_{5i})$.

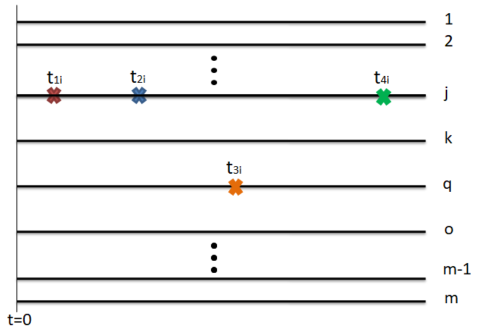
Since the socket index does not matter (we have the i.i.d. assumption), the situations 2 and 6, the situations 3, 4, 7 and 8, the situations 9, 10 and 11 and the situations 12 and 13 present the same contribution for the conditional distribution of $f(d_{5i} \mid \boldsymbol{\mathcal{T}}_i, d_{4i}, d_{3i}, d_{2i}, d_{1i})$.



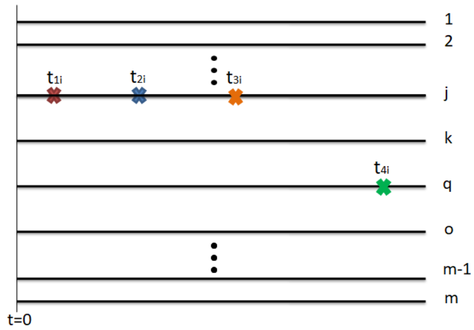
(a) Situation 1



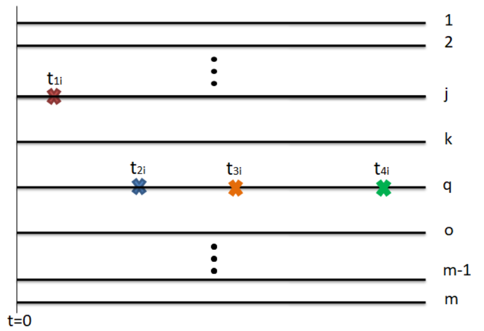
(b) Situation 2



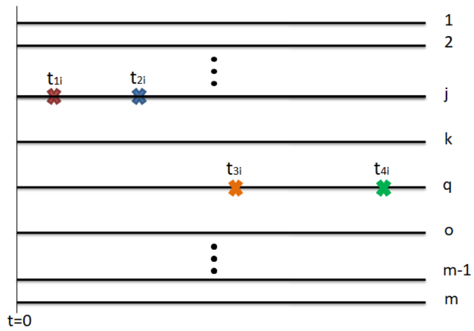
(c) Situation 3



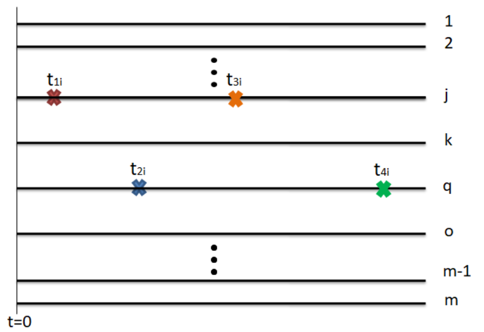
(d) Situation 4



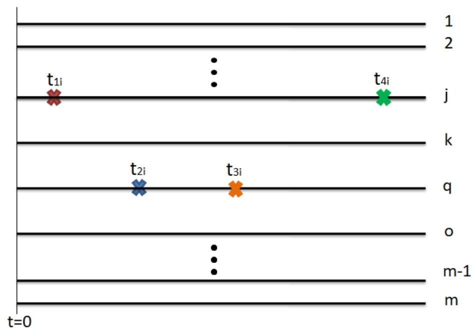
(e) Situation 5



(f) Situation 6

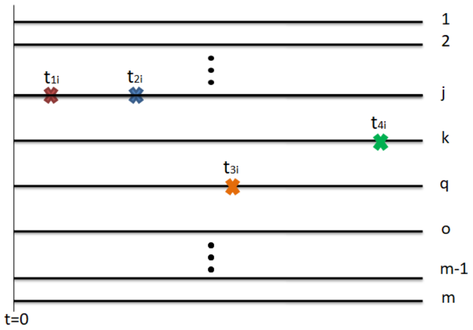


(g) Situation 7

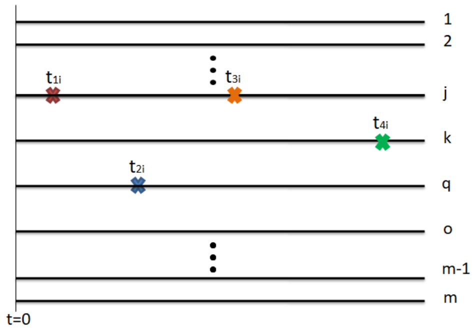


(h) Situation 8

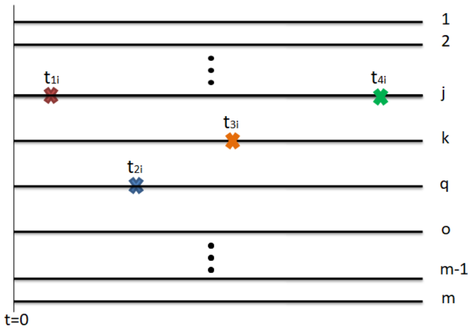
Figure 3: The fifteen possible situations of how four failures can occur in the sockets - 1 to 8.



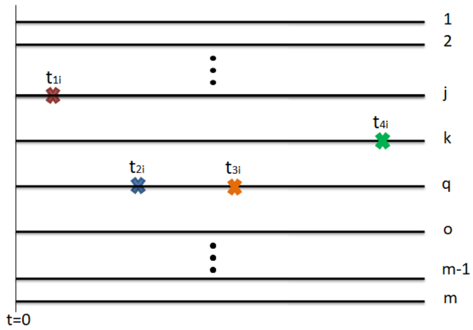
(a) Situation 9



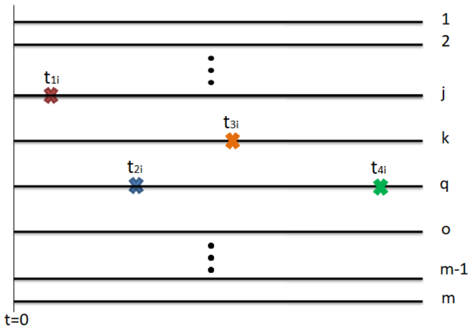
(b) Situation 10



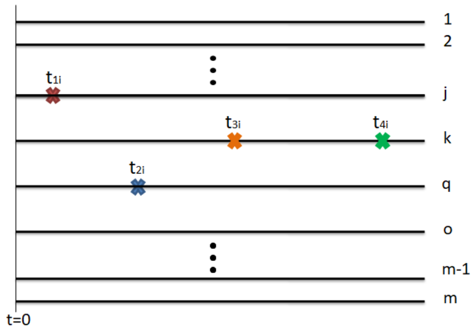
(c) Situation 11



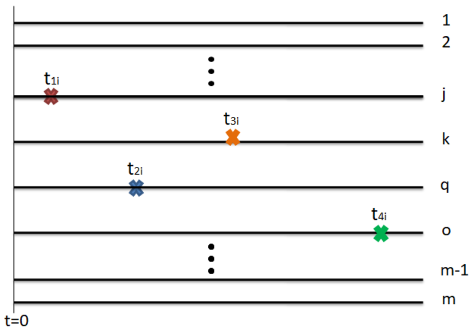
(d) Situation 12



(e) Situation 13



(f) Situation 14



(g) Situation 15

Figure 4: The fifteen possible situations of how four failures can occur in the sockets - 9 to 15.

2 Study of the number of Monte Carlo sample

In order to study the impact of the number of Monte Carlo sample on the maximum likelihood estimator via EM algorithm (EM-ML), we conducted the simulations for all combinations of the following features: $n \in \{10, 50, 100, 200\}$, $m \in \{4, 8, 16, 32\}$, and $m_c \in \{4, 8\}$, resulting in 32 scenarios, in which n is the sample size, m is the number of sockets, and m_c is censoring mean lifetime. For more details about how dataset are generated can be found at Algorithm 2 in the manuscript.

For each scenario, a dataset was generated, and we compare the mean absolute error (MAE) from the estimator resulting from EM-ML to the true distribution. We consider 19 values of Monte Carlo sample, from $L = 100$ to $L = 1000$ with a gap of 50 units, that is, $L \in \{100, 150, 200, \ldots, 900, 950, 1000\}$.

As one can see in Tables 1 and 2 for $m_c = 4$, and in Tables 3 and 4 for $m_c = 8$, the MAE value is practically the same regardless of the value of L for all scenarios considered. For this reason, we choose $L = 100$.

Table 1: MAE obtained for different values of L by considering $m_c = 4$ and the sample sizes $n = 10$ and $n = 50$.

n=10				
L values	m = 4	m = 8	m = 16	m = 32
100	0.097054958	0.175500734	0.039017419	0.028261852
150	0.097054958	0.175500734	0.041307128	0.028831198
200	0.097054958	0.175500734	0.041169920	0.028717485
250	0.097054958	0.175500734	0.039930794	0.028359780
300	0.097054958	0.175500734	0.041254209	0.028374573
350	0.097054958	0.175500734	0.041829177	0.028333045
400	0.097054958	0.175500734	0.042520935	0.028383278
450	0.097054958	0.175500734	0.041046871	0.028050760
500	0.097054958	0.175500734	0.040677652	0.027863253
550	0.097054958	0.175500734	0.041160202	0.028546671
600	0.097054958	0.175500734	0.042929532	0.028773057
650	0.097054958	0.175500734	0.040636138	0.028894523
700	0.097054958	0.175500734	0.041199316	0.028191647
750	0.097054958	0.175500734	0.042168413	0.028594575
800	0.097054958	0.175500734	0.042168460	0.028466352
850	0.097054958	0.175500734	0.040922980	0.028375690
900	0.097054958	0.175500734	0.041212946	0.028320464
950	0.097054958	0.175500734	0.041324984	0.028865522
1000	0.097054958	0.175500734	0.040823293	0.028215566
n=50				
L values	m = 4	m = 8	m = 16	m = 32
100	0.052503099	0.026746586	0.071195075	0.030188679
150	0.052484352	0.027169478	0.070917475	0.031274096
200	0.053104306	0.027155158	0.070820876	0.031520007
250	0.052648328	0.028089157	0.071161903	0.031300216
300	0.053291119	0.026710387	0.070967350	0.031454286
350	0.052314070	0.027362181	0.070925535	0.031315114
400	0.052699837	0.026725353	0.070944947	0.031237881
450	0.052854499	0.027266589	0.070850462	0.031252745
500	0.053047834	0.027209117	0.070941087	0.031160991
550	0.052579583	0.026948179	0.070931525	0.031308010
600	0.052795520	0.027135105	0.070882863	0.031299577
650	0.053333066	0.027405817	0.071030530	0.031359314
700	0.052253845	0.027100874	0.070852965	0.031320288
750	0.052950862	0.027200043	0.070875773	0.031192603
800	0.053211232	0.027326351	0.070891527	0.031334819
850	0.053364006	0.027208229	0.070795752	0.031397174
900	0.052755141	0.027153171	0.070705550	0.031139484
950	0.052739592	0.027362893	0.070992356	0.031270248
1000	0.052739873	0.026895496	0.071079454	0.031334819

Table 2: MAE obtained for different values of L by considering $m_c = 4$ and the sample sizes $n = 100$ and $n = 200$.

n=100				
L values	m = 4	m = 8	m = 16	m = 32
100	0.022839142	0.014172490	0.057617283	0.054777789
150	0.023216644	0.014246455	0.057392279	0.054872264
200	0.023502981	0.014523670	0.057753581	0.054999646
250	0.023754074	0.014201453	0.057617669	0.054802554
300	0.023559074	0.014320741	0.057553859	0.054862170
350	0.023578610	0.014242770	0.057668731	0.054738523
400	0.023347806	0.014268691	0.057623194	0.054983885
450	0.023444192	0.014186468	0.057623194	0.054880744
500	0.023426798	0.014364931	0.057665162	0.054930038
550	0.023540226	0.014316386	0.057530328	0.054684366
600	0.023589652	0.014277114	0.057688006	0.054942658
650	0.023393762	0.014371262	0.057567588	0.054858216
700	0.023549651	0.014304628	0.057561496	0.054833598
750	0.023670828	0.014266177	0.057607110	0.054852143
800	0.023583018	0.014327204	0.057509242	0.054788753
850	0.023724339	0.014349209	0.057621745	0.054894436
900	0.023464710	0.014268768	0.057513417	0.054846108
950	0.023725148	0.014302161	0.057621745	0.054770109
1000	0.023612624	0.014422488	0.057668731	0.054880459
n=200				
L values	m = 4	m = 8	m = 16	m = 32
100	0.068695347	0.049739792	0.040244256	0.051742457
150	0.068562170	0.049702862	0.040017403	0.051670821
200	0.068780031	0.049392601	0.040054197	0.051536128
250	0.068619219	0.049449440	0.040339403	0.051700190
300	0.068745562	0.049509093	0.040110908	0.051700190
350	0.068455005	0.049468688	0.040035246	0.051616674
400	0.068745656	0.049566532	0.040067145	0.051615889
450	0.068570272	0.049617624	0.040127401	0.051615889
500	0.068618198	0.049565727	0.039979232	0.051655221
550	0.068658518	0.049593786	0.040110908	0.051716213
600	0.068706245	0.049606438	0.040099271	0.051615889
650	0.068472913	0.049628929	0.040099271	0.051716213
700	0.068814244	0.049476902	0.040109573	0.051716213
750	0.068707035	0.049628929	0.040032661	0.051648285
800	0.068685085	0.049364919	0.040214856	0.051750504
850	0.068690163	0.049430072	0.039974270	0.051652866
900	0.068707758	0.049537663	0.040099271	0.051615889
950	0.068701848	0.049393938	0.040148613	0.051655640
1000	0.068533125	0.049488026	0.040077462	0.051700190

Table 3: MAE obtained for different values of L by considering $m_c = 8$ and the sample sizes $n = 10$ and $n = 50$.

n=10				
L values	m = 4	m = 8	m = 16	m = 32
100	0.055121580	0.016789957	0.019414837	0.026154745
150	0.055082742	0.016860055	0.020463381	0.026567874
200	0.054444113	0.015864028	0.018835251	0.026462550
250	0.055069380	0.016713569	0.019871740	0.026594106
300	0.054485073	0.016367850	0.019853507	0.026738133
350	0.055189594	0.017587452	0.019448385	0.026237730
400	0.055019006	0.015934872	0.019920420	0.026875568
450	0.054713186	0.016490724	0.019934674	0.026280418
500	0.055189288	0.016273133	0.019732631	0.026681412
550	0.054836997	0.016014315	0.019825123	0.026455151
600	0.054858070	0.016428943	0.019595581	0.026533181
650	0.054995108	0.016428895	0.019664885	0.026564603
700	0.054629365	0.016489076	0.019735842	0.026029188
750	0.054856907	0.016709172	0.019832785	0.026806468
800	0.054775048	0.016788874	0.019521586	0.026574314
850	0.055063353	0.016686573	0.019903785	0.026610019
900	0.054661334	0.016651362	0.019802436	0.026478461
950	0.054942532	0.016801348	0.019524561	0.026358301
1000	0.054737599	0.016525393	0.019627528	0.026316266
n=50				
L values	m = 4	m = 8	m = 16	m = 32
100	0.012389634	0.014378597	0.016346480	0.005987124
150	0.012845588	0.013487204	0.016044449	0.006514487
200	0.012529827	0.013781535	0.016310423	0.006309129
250	0.012643199	0.013410669	0.016438114	0.006138570
300	0.012626224	0.014027554	0.016211230	0.006317077
350	0.012565908	0.013624478	0.016106788	0.006408499
400	0.012497260	0.013582925	0.016144317	0.006228219
450	0.012549399	0.013530021	0.016143322	0.006086537
500	0.012367086	0.013401940	0.016141858	0.006349109
550	0.012478901	0.013735523	0.016148057	0.006133311
600	0.012575963	0.013727710	0.015924829	0.006150113
650	0.012406790	0.013679476	0.015832550	0.006291790
700	0.012361329	0.013702765	0.016065679	0.006370769
750	0.012591359	0.013581635	0.016056227	0.006241175
800	0.012627465	0.013626546	0.016348414	0.006344578
850	0.012589741	0.013679997	0.016154883	0.006315521
900	0.012400539	0.013581913	0.016144224	0.006270848
950	0.012714286	0.013548530	0.016227546	0.006106073
1000	0.012525310	0.013373894	0.015997688	0,006328456

Table 4: MAE obtained for different values of L by considering $m_c = 8$ and the sample sizes $n = 100$ and $n = 200$.

n=100				
L values	m = 4	m = 8	m = 16	m = 32
100	0.017478142	0.023298946	0.011639431	0.007226465
150	0.017504665	0.023831008	0.011615005	0.007020751
200	0.017347558	0.023896156	0.011965275	0.007119573
250	0.017055132	0.023697194	0.011933227	0.007241848
300	0.017297420	0.023730095	0.011731473	0.007061526
350	0.017517888	0.023597261	0.011820039	0.007203191
400	0.017479327	0.023599011	0.011677016	0.007240931
450	0.017317435	0.023874208	0.011614493	0.007141878
500	0.017059635	0.023509867	0.011812953	0.007206306
550	0.017072872	0.024041694	0.011687009	0.007202986
600	0.017148568	0.023595875	0.011809882	0.007229320
650	0.017190565	0.023723113	0.011641495	0.007084835
700	0.017188677	0.023643264	0.011745399	0.007153466
750	0.017168542	0.023465625	0.011816846	0.007170712
800	0.017249045	0.023787240	0.011667448	0.007224234
850	0.017385527	0.023708325	0.011784952	0.007184772
900	0.017383595	0.023508001	0.011732255	0.007203166
950	0.017193765	0.023460047	0.011758893	0.007148787
1000	0.017230682	0.023610863	0.011769666	0.007163527
n=200				
L values	m = 4	m = 8	m = 16	m = 32
100	0.009805856	0.008268677	0.009024516	0.007869445
150	0.010201032	0.008359005	0.008951202	0.007902837
200	0.009806975	0.008283252	0.009280869	0.007889820
250	0.009733988	0.008287597	0.009163627	0.007768234
300	0.009604532	0.008237118	0.009182839	0.007852535
350	0.009617489	0.008294073	0.009226847	0.007886357
400	0.009839392	0.008272793	0.009203195	0.007829124
450	0.009801435	0.008273763	0.009196935	0.007859767
500	0.009811608	0.008268882	0.009301848	0.007897819
550	0.009796365	0.008308688	0.009194947	0.007747985
600	0.009628443	0.008306949	0.009229040	0.007822580
650	0.009646638	0.008270202	0.009222904	0.007815548
700	0.009464440	0.008293359	0.009288022	0.007798927
750	0.009617270	0.008279542	0.009215189	0.007807101
800	0.009672819	0.008311923	0.009211411	0.007888251
850	0.009746441	0.008296962	0.009151720	0.007820409
900	0.009713779	0.008281549	0.009284245	0.007810474
950	0.009719215	0.008283652	0.009285797	0.007802866
1000	0.009829891	0.008285506	0.009201503	0.007792199