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Visible points, the separation problem, and applications to MINLP

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Visible points, the separation problem, and applications to MINLP

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Abstract

In this paper we introduce a technique to produce tighter cutting planes for mixed-integer non-linear programs. Usually, a cutting plane is generated to cut off a specific infeasible point. The underlying idea is to use the infeasible point to restrict the feasible region in order to obtain a tighter domain. To ensure validity, we require that every valid cut separating the infeasible point from the restricted feasible region is still valid for the original feasible region. We translate this requirement in terms of the separation problem and the reverse polar. In particular, if the reverse polar of the restricted feasible region is the same as the reverse polar of the feasible region, then any cut valid for the restricted feasible region that *separates* the infeasible point, is valid for the feasible region.

We show that the reverse polar of the *visible points* of the feasible region from the infeasible point coincides with the reverse polar of the feasible region. In the special where the feasible region is described by a single non-convex constraint intersected with a convex set we provide a characterization of the visible points. Furthermore, when the non-convex constraint is quadratic the characterization is particularly simple. We also provide an extended formulation for a relaxation of the visible points when the non-convex constraint is a general polynomial.

Finally, we give some conditions under which for a given set there is an inclusion-wise smallest set, in some predefined family of sets, whose reverse polars coincide.

Keywords: Separation problem, Visible points, Mixed-integer non-linear programming, Reverse polar, Global optimization.

1 Introduction

The separation problem is a fundamental problem in optimization [7]. Given a set $S \subseteq \mathbb{R}^n$ and a point $\bar{x}$, the separation problem is

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Decide if $\bar{x} \in \overline{\text{conv}}(S)$ or find an inequality $\alpha x \leq \beta$ that separates $\bar{x}$ from $\overline{\text{conv}}(S)$.

Algorithms to solve optimization problems, especially those based on solving relaxations, such as branch and bound, need to deal with the separation problem. Consider, for example, solving a mixed integer linear problem via branch and bound [3, Section 9.2]. The solution to the linear relaxation plays the role of $\bar{x}$, while a relaxation based on a subset of the constraints is used as S for the separation problem, see [3, Chapter 6].

The separation problem can be rephrased in terms of the *reverse polar* [1, 17] of S at $\bar{x}$, defined as

$$S^{\bar{x}} = \{\alpha \in \mathbb{R}^n : \alpha^{\mathsf{T}}(x - \bar{x}) \geq 1, \forall x \in S\}.$$

The elements of $S^{\bar{x}}$ are the normals of the hyperplanes that separate $\bar{x}$ from $\overline{\text{conv}}(S)$. Hence, the separation problem can be stated equivalently as

Decide if $S^{\bar{x}}$ is empty or find an element from it.

The point of departure of the present work is the following observation.

Observation 1. If there is a set V such that $(S \cap V)^{\bar{x}} = S^{\bar{x}}$, then, as far as the separation problem is concerned, the feasible region can be regarded as $S \cap V$ instead of S.

A set V such that $V^{\bar{x}} = S^{\bar{x}}$ will be called a *generator* of $S^{\bar{x}}$. Intuitively, if a set V is such that $V \cap S$ generates $S^{\bar{x}}$, that is, if we can ensure that a cut valid for $V \cap S$ that separates $\bar{x}$ is also valid for S, then V should at least contain the points of S that are "near" $\bar{x}$. To formalize the meaning of "near" we use the concept of *visible points* [5] of S from $\bar{x}$, which are the points $x \in S$ for which the segment joining x with $\bar{x}$ only intersects S at x, see Definition 4. In other words, they are the points of S that can be "seen" from $\bar{x}$. In Proposition 8 we show that the visible points are a generator of $S^{\bar{x}}$.

As a motivation, we present an application of our results in the context of nonlinear programming, which is treated in more detail in Section 4.

Example 1. Consider the separation problem of $\bar{x} = (0, 0)$ from $S = \{x \in B : g(x) \leq 0\}$ where

$$B = [-\tfrac{1}{2}, 3] \times [-\tfrac{1}{2}, 3],$$
$$g(x_1, x_2) = -x_1^2 x_2 + 5x_1 x_2^2 - x_2^2 - x_2 - 2x_1 + 2,$$

as depicted in Figure 1. A standard technique for solving the separation problem for S and $\bar{x}$ is to construct a convex underestimator of g over B [16, Sections 6.1.2 and 7.5.1]. The quality of a convex underestimator depends on the bounds of the variables and tighter bounds yield tighter underestimators. As we will see (Proposition 8 and Lemma 23), $R^{\bar{x}} = S^{\bar{x}}$ where

$$R = \{x \in B : g(x) = 0, \langle \nabla g(x), x \rangle \leq 0\}.$$

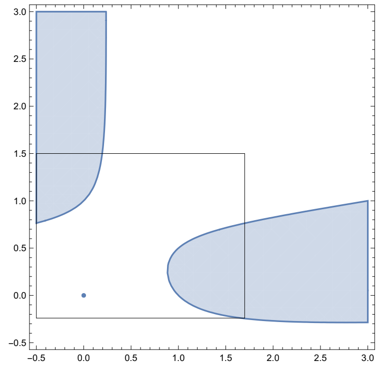

Figure 1: The feasible region $g(x) \leq 0$ and $\bar{x} = (0, 0)$ together with the box V.

It is possible to show that $R \subseteq V$, where $V = [-\frac{1}{2}, \frac{17}{10}] \times [-\frac{6}{25}, \frac{3}{2}]$. Hence, by Corollary 21, $(V \cap S)^{\bar{x}} = S^{\bar{x}}$. This means that we can solve the separation problem over $\{x \in V : g(x) \leq 0\}$ instead of S. Therefore, if we were to compute an underestimator of g, it could be computed over $V \subsetneq B$. $\square$

Methods for obtaining tighter bounds for mixed integer nonlinear programming (MINLP) are of paramount importance. Indeed, not only bound tightening procedures enhance the performance of MINLP solvers, but also many algorithms for solving MINLPs require that all variables are bounded [8]. We refer to the recent survey [11] for more information on bound tightening procedures and its impact on MINLP solvers, and to [2] for the practical importance of MINLP.

However, the technique that we introduce in this paper is *not* a bound tightening technique in the classic sense, i.e., the tighter bounds that might be learned from V are not valid for the original problem, but only for the separation problem at hand.

We would like to point out that in [15] a similar idea — to modify the separation problem — is used in the context of stochastic mixed integer programming. The objective of the authors of [15] is to speed-up the solution of the separation problem. In contrast, our objective is to produce tighter cutting planes for MINLP.

Contributions We show that for every closed set S, there exists an inclusion-wise smallest closed convex set that generates $S^{\bar{x}}$ (Theorem 17). When S is compact, there is an inclusion-wise smallest closed set that generates $S^{\bar{x}}$ (Theorem 19). Furthermore, under some mild assumptions on S, we show that there is an inclusion-wise smallest closed convex set C such that $C \cap S$ generates $S^{\bar{x}}$ (Theorem 18). We also show the existence of a generator, $V_S(\bar{x})$, of $S^{\bar{x}}$ which is more suitable for computations.

We apply our results to MINLP and give an explicit description of $V_S(\bar{x})$ when $S = \{x \in C : g(x) \leq 0\}$, where C is a closed convex set containing $\bar{x}$, and g is continuous (Section 4.1). For the important case of quadratic constraints, i.e.,

when g is a quadratic function, we show that $V_S(\bar{x})$ has a particularly simple expression (Theorem 25).
For the case when g is a general polynomial, we provide an extended formulation for a relaxation of $V_S(\bar{x})$ based on the theory of non-negative univariate polynomials (Theorem 29).

Definitions and notation Given a set S, $\operatorname{conv} S, \operatorname{cl} S, \overline{\operatorname{conv}}\, S, \operatorname{ext} S$ represent the convex hull, the closure, the closure of the convex hull and the extreme points of S, respectively. The *extreme points* of a, not necessarily convex, set S are the points in S that cannot be written as convex combination of others. Given some set $S = \{(x, y) : \ldots\}$, we use $\operatorname{proj}_x S$ to denote the projection of S to the x-space, that is, $\operatorname{proj}_x S = \{x : \exists y, (x, y) \in S\}$. If $g : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function and $D \subseteq C$, then we denote by g_D^{vex} a convex underestimator of g over D. When g is convex, $\partial g(x)$ denotes the subdifferential of g at x.
Given an interval $I \subseteq \mathbb{R}$ and an arbitrary set $A \subseteq \mathbb{R}^n$ we denote by IA the set $\{\lambda x : \lambda \in I, x \in A\}$. Likewise, for $x \in \mathbb{R}^n$, $Ix := \{\lambda x : \lambda \in I\}$. Given an integer d, we denote by $\mathcal{S}_+^d$ the cone of positive semi-definite matrices of size d. Finally, we use interchangeably the dot product notation $\langle c, x\rangle$ and $c^\mathsf{T} x$.

2 Visible points and the reverse polar

In this section we introduce the concept of visible points and reverse polar, and state some basic properties about them, which we will use in the rest of the paper. The main result in this section is that the reverse polar of the visible points of a set is the reverse polar of the set (Proposition 8).

Unless stated otherwise, we will assume $\bar{x} = 0$. This is without loss of generality, since we can always translate the set S to $S - \bar{x}$. We start by restating the definition of reverse polar.

Definition 2. Let $S \subseteq \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$. The *reverse polar* of S at $\bar{x}$ is

$$S^{\bar{x}} = \{\alpha \in \mathbb{R}^n : \alpha^\mathsf{T}(x - \bar{x}) \geq 1, \text{ for all } x \in S\}.$$

As stated in the introduction, the reverse polar contains all cuts that separate $\bar{x}$ from S.

Definition 3. Let $S, V \subseteq \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$. We say that V is a *generator* of $S^{\bar{x}}$ if and only if

$$V^{\bar{x}} = S^{\bar{x}}$$

Definition 4. Let $S \subseteq \mathbb{R}^n$ be closed and $\bar{x} \notin S$. The set of *visible points* of S from $\bar{x}$ is

$$\begin{aligned} V_S(\bar{x}) &= \{x \in S : (x + [0, 1](\bar{x} - x)) \cap S = \{x\}\} \\ &= \{x \in S : (x + (0, 1](\bar{x} - x)) \cap S = \emptyset\}. \end{aligned}$$

We denote $V_S(0)$ by V_S and note that

$$V_S = \{x \in S : [0, 1]x \cap S = \{x\}\} = \{x \in S : [0, 1)x \cap S = \emptyset\}.$$

The following concept is, in some sense, the opposite of the visible points.

Definition 5. Let $S \subseteq \mathbb{R}^n$ be closed. The *shadow* of S from 0 is

$$\operatorname{shw} S = [1, \infty) S.$$

The concept of shadow has also been called *penumbra* [12, p. 22],[14, 4] and *aureole closure* [13]. The following are some basic properties of the reverse polar.

Lemma 6. *[13, Property 9.2.2] Let $S, T \subseteq \mathbb{R}^n$. Then,*

1. $S^0 = (\operatorname{shw} S)^0 = (\operatorname{conv} S)^0 = (\operatorname{cl} S)^0$.
2. $S^0 = \emptyset$ *if and only if* $0 \in \overline{\operatorname{conv}} S$.
3. $S \subseteq T$ *implies* $T^0 \subseteq S^0$.
4. *If* $0 \notin \overline{\operatorname{conv}} S$*, then* $(S^0)^0 = \operatorname{shw} \overline{\operatorname{conv}} S$.

We will now show that V_S is a generator of S^0. To this end, we need the following lemma, which says that the shadow of what can be seen of a set is the same as the shadow of the whole set. Likewise, what can be seen of a set is the same as what can be seen of the shadows of the set.

Lemma 7. *Let $S \subseteq \mathbb{R}^n$ be a closed set such that $0 \notin S$. Then,* $\operatorname{shw} V_S = \operatorname{shw} S$ *and* $V_{\operatorname{shw} S} = V_S$.

Proof. First we prove that $\operatorname{shw} V_S = \operatorname{shw} S$. Clearly, $\operatorname{shw} V_S \subseteq \operatorname{shw} S$.

Let $y \in \operatorname{shw} S$, then $y = \lambda x$ with $x \in S$, $\lambda \geq 1$. Let $I = \{\mu \geq 0 : \mu x \in S\}$ and $\mu_0 = \min I$. The minimum exists since I is closed and not empty as S is closed and $1 \in I$, respectively. From $1 \in I$, we deduce $\mu_0 \leq 1$, and from $0 \notin S$, $\mu_0 > 0$. Hence, $\mu_0 x \in V_S$ and $y = \frac{\lambda}{\mu_0}(\mu_0 x) \in \operatorname{shw} V_S$, since $\frac{\lambda}{\mu_0} \geq 1$.

Now we prove that $V_{\operatorname{shw} S} = V_S$. Clearly, $S \subseteq \operatorname{shw} S$ implies that $V_S \subseteq V_{\operatorname{shw} S}$.

Let $x_0 \in V_{\operatorname{shw} S}$. Then $x_0 \in \operatorname{shw} S$, so there exists $\lambda \geq 1$ and $x \in S$ such that $x_0 = \lambda x$. Note that $\lambda = 1$, since otherwise, $\frac{1}{\lambda} x_0 = x \in S \subseteq \operatorname{shw} S$ which cannot be as x_0 is visible. Thus, $x_0 \in V_S$. □

Proposition 8. *Let $S \subseteq \mathbb{R}^n$ be a closed set. Then,*

$$(S \cap V_S)^0 = V_S^0 = S^0.$$

Proof. The first equality just comes from the fact that $V_S \subseteq S$.

If $0 \in S$, then the equality holds as all the sets are empty. Otherwise, the equality follows from $V_S^0 = (\operatorname{shw} V_S)^0 = (\operatorname{shw} S)^0 = S^0$, where the first and last equalities are by Lemma 6 and the middle one, by Lemma 7. □

3 The smallest generators

3.1 Motivation

In the previous section we showed that there is a set $U \subseteq S$ such that $(U \cap S)^0 = S^0$, namely, $U = V_S$. This set can be used to improve separation routines as was shown already in Example 1. We will come back to applications of the visible points to separation in the next section.

The topic of this section is motivated by the following example, where the set V_S is much larger than the smallest generator.

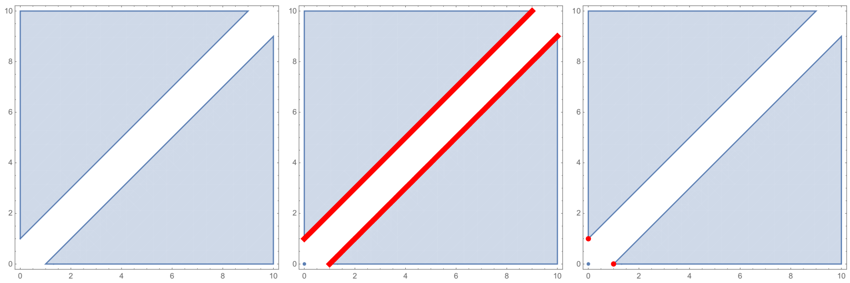

Figure 2: The region S. In the middle picture V_S are the points described by the thick red line. In the right picture the red points form the smallest set V such that $V^0 = S^0$.

Example 2. Consider the constrained set $S = \{(x_1, x_2) \in \mathbb{R}^2_+ \,:\, (x_1 - x_2)^2 \geq 1\}$ depicted in Figure 2. The visible points are the lines $x_2 = x_1 + 1$ and $x_2 = x_1 - 1$ intersected with the first orthant. However, it is not hard to see that $V = \{(0, 1), (1, 0)\}$ is the smallest *closed* generator of S^0. □

This example motivates the following question.

Question 9. What is, if any, the smallest closed set U such that $U^0 = S^0$?

The reason we restrict to generators that are closed sets is to avoid representation issues. For example, if S is the ball of radius 1 centered at $(2, 0)$, then Theorem 25 implies that the left arc joining $(2, 1)$ and $(2, -1)$ generates S^0. However, the rational points on this arc also generate S^0 and the smallest set generating S^0 does not exist. In order to avoid such issues, we concentrate on closed generators.

As can be seen from simple examples, such as $S = \mathbb{R}_+ \times \{1\}$ for which every $a \geq 0$ defines the generator $(\{0\} \cup [a, \infty)) \times \{1\}$, the smallest closed generator must not exist. However, a smallest closed convex generator might exist and so we ask the following question.

Question 10. What is, if any, the smallest closed convex generator of S^0?

We are mainly interested in applying our results to the separation problem, as already explained in the introduction. In that case, the set S usually looks like $S = C \cap F$, where C is a convex set and F is the sublevel set of some non-convex function, see the next section. In this context, replacing C by a smaller convex set might be beneficial for the separation problem (see Example 5). Thus, it is also natural to consider the following question.

Question 11. What is, if any, the smallest closed convex set U such that $S \cap U$ generates S^0?

The last two questions are not the same. Informally, S is only used to define S^0 in Question 10, and so any other set T such that $T^0 = S^0$ can be used to formulate the question. For instance, we can assume without loss of generality that S is closed and convex, since Lemma 6 implies that $(\overline{\text{conv}}\, S)^0 = S^0$. In contrast, in Question 11 we are asking for the smallest generator contained in S.

As we will see, the answer to Question 10 is that $\overline{\text{conv}}\, V_{\overline{\text{conv}}\, S}$ is the smallest closed convex generator of S^0. However, the next two examples show that Question 11 is a bit more delicate.

The first example shows that, in general, there is no unique smallest closed convex set U such that $(S \cap U)^0 = S^0$.

Example 3. Let $S = \{(1,0), (0,1), (-1,0), (0,-1)\}$. Since $0 \in \text{conv}\, S$, $S^0 = \emptyset$.

Clearly $V = \{0\} = V_{\text{conv}\, S}$ is the smallest closed convex set such that $V^0 = \emptyset$. However, $S \cap V = \emptyset$, which implies that $(S \cap V)^0 = \mathbb{R}^2 \neq S^0$. Furthermore, $U_1 = \{(\lambda, 0) : \lambda \in [-1,1]\}$ and $U_2 = \{(0,\lambda) : \lambda \in [-1,1]\}$ are both closed convex and $(U_i \cap S)^0 = S^0$. Since $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$ we conclude that there is no smallest closed convex set U such that $(U \cap S)^0 = S^0$. □

However, we cannot even expect to find a minimal closed convex set U such that $(S \cap U)^0 = S^0$.

Example 4. Let $S = \{(0,1)\} \cup \{(\lambda, 2) : \lambda \geq 0\}$. We have $S^0 = \{\alpha : \alpha_1 \geq 0, \alpha_2 \geq 1\}$.

Indeed, $(0,1) \in S$ implies that $\alpha_2 \geq 1$. If $\alpha_1 < 0$ for some $\alpha \in S^0$, then there is a large enough λ such that $\lambda\alpha_1 + 2\alpha_2 < 1$ and $(\lambda, 2) \in S$. On the other hand, if $\alpha_1 \geq 0$ and $\alpha_2 \geq 1$, then $\alpha_1 x_1 + \alpha_2 x_2 \geq 1$ for every $(x,y) \in S$.

Let $T_M = \{(0,1)\} \cup \{(\lambda, 2) : \lambda \geq M\}$ and $U_M = \overline{\text{conv}}(T_M)$. The same argument as above shows that $(U_M \cap S)^0 = T_M^0 = S^0$. Notice that any U with $(U \cap S)^0 = S^0$ must contain a sequence $\lambda_n \to \infty$ such that $(\lambda_n, 2) \in S$. Thus, any minimal U, if it exists, must be of the form U_M for some $M \geq 0$.

It is clear that $U_{M_1} \subseteq U_{M_2}$ if and only if $M_1 > M_2$ and $\bigcap_{M>0} U_M = \{(\lambda, 1) : \lambda \geq 0\}$. However, $S \cap \{(\lambda, 1) : \lambda \geq 0\} = \{(0,1)\}$ and $\{(0,1)\}^0 \neq S^0$. Therefore, there is no minimal U.

On the other hand, $V = \{(\lambda, 1) : \lambda \geq 0\} = V_{\overline{\text{conv}}\, S}$ is the smallest closed convex set such that $V^0 = S^0$. □

However, these are the only "pathological cases". Indeed, as we will see, if $\text{conv}(S)$ is closed (e.g. when S is compact) and $0 \notin \text{conv}\, S$, (i.e., $S^0 \neq \emptyset$), then $\overline{\text{conv}}\, V_{\text{conv}\, S}$ is the smallest closed convex set such that $\overline{\text{conv}}\, V_{\text{conv}\, S} \cap S$ generates S^0.

Remark 12. The closure operations are needed because, in general, V_S and $\text{conv}\, V_S$ are not closed, even when S is convex and compact. Indeed, it is shown in [5, Example 15.5] that for

$$S := (1, 0, 0) + \text{cone}\{(1, \alpha, \beta) : \alpha^2 + (\beta - 1)^2 \leq 1\},$$

V_S is open. The authors show that the points $(2, \sin(t), 1 + \cos(t))$ are visible for $t \in (0, \pi)$, but the limit when t approaches π, (2,0,0), is not. The remark follows from a modification of this example so that S is compact, e.g., by intersecting it with $[0, 3] \times \mathbb{R}^2$.

3.2 Preliminaries

Here we collect a few lemmata that we are going to need in order to answer Questions 9, 10 and 11.

Lemma 13. *[5, Proposition 15.19] Let S be a closed convex set such that $0 \notin S$. If $x \in V_S$ is a strict convex combination of $x_1, \ldots x_m \in S$, then $x_1, \ldots x_m \in V_S$.*

This result immediately implies the following two lemmata.

Lemma 14. *Let $S \subseteq \mathbb{R}^n$ be a closed convex set such that $0 \notin S$. Then, $\text{ext}\, V_S = V_S \cap \text{ext}\, S$.*

Proof. We start by proving $\text{ext}\, V_S \subseteq V_S \cap \text{ext}\, S$. Let $x \in \text{ext}\, V_S$. Clearly, $x \in V_S$. If $x \notin \text{ext}\, S$, then there are $x_1, \ldots, x_m \in S$ such that x is a strict convex combination of $x_1, \ldots, x_m$. Lemma 13 implies that $x_i \in V_S$ for every $i = 1, \ldots, m$. Thus, x is not an extreme point of V_S. This contradiction proves that $x \in \text{ext}\, S$.

If $x \in V_S \cap \text{ext}\, S$ but $x \notin \text{ext}\, V_S$, then x is a strict convex combination of some elements of V_S. Since $V_S \subseteq S$, x is a strict convex combination of some element of S. This is a contradiction with $x \in \text{ext}\, S$. □

Lemma 15. *Let $S \subseteq \mathbb{R}^n$ be closed set such that $\text{conv}\, S$ is closed and $0 \notin \text{conv}\, S$. Then,*

$$\text{conv}(V_{\text{conv}\, S}) = \text{conv}(S \cap V_{\text{conv}\, S}).$$

Proof. From $S \cap V_{\text{conv}\, S} \subseteq V_{\text{conv}\, S}$, it follows that $\text{conv}(S \cap V_{\text{conv}\, S}) \subseteq \text{conv}(V_{\text{conv}\, S})$.

To prove the other inclusion it is enough to show that $V_{\operatorname{conv} S} \subseteq \operatorname{conv}(S \cap V_{\operatorname{conv} S})$. Let $x \in V_{\operatorname{conv} S}$. Then, $x \in \operatorname{conv}(S)$ and so x is a strict convex combination of some points of $x_1, \ldots, x_m \in S$. Then, by Lemma 16, $x_1, \ldots, x_m \in S \cap V_{\operatorname{conv} S}$. Thus, $x \in \operatorname{conv}(S \cap V_{\operatorname{conv} S})$. □

We remark that the previous lemma does not follow from Lemma 14 by just taking the convex hull operation to the equality, since $\operatorname{conv} S$ may not have extreme points.

The following is a slight extension of [12, Corollary 18.3.1].

Lemma 16. *Let $S \subseteq \mathbb{R}^n$ be a closed set. Then,* $\operatorname{ext} \overline{\operatorname{conv}}\, S \subseteq S$.

Proof. Recall that x_0 is an exposed point [12, Section 18] of a closed convex set C if and only if there exists an α such that $\{x_0\} = \arg\max_{x \in C} \alpha^\mathsf{T} x_0$.

We will show that the exposed points of $\overline{\operatorname{conv}}\, S$ is a subset of S. Then, by Straszewicz's Theorem [12, Theorem 18.6] and the closedness of S, it follows that $\operatorname{ext} \overline{\operatorname{conv}}\, S \subseteq S$. Note that when the set of exposed points is empty, the result follows trivially. Thus, we assume that the set of exposed points is non-empty.

Let x_0 be an exposed point of $\overline{\operatorname{conv}}\, S$ and let α be a direction that exposes it. Then, $\sup_{x \in S} \alpha^\mathsf{T} x = \alpha^\mathsf{T} x_0$. Since S is closed, there exists $x_1 \in S$ such that $\alpha^\mathsf{T} x_1 = \alpha^\mathsf{T} x_0$. However, since $x_1 \in S \subseteq \overline{\operatorname{conv}}\, S$ and α exposes x_0, we must have $x_1 = x_0$. Thus, $x_0 \in S$. □

3.3 Results

Let us start by answering Question 10.

Theorem 17. *Let $S \subseteq \mathbb{R}^n$ be closed. Then,*

$$(\overline{\operatorname{conv}}\, V_{\overline{\operatorname{conv}}\, S})^0 = S^0.$$

Furthermore, if $C \subseteq \mathbb{R}^n$ is a closed convex generator of S^0, then

$$\overline{\operatorname{conv}}(V_{\overline{\operatorname{conv}}\, S}) \subseteq C.$$

Proof. Note that if $S^0 = \emptyset$, then $0 \in \overline{\operatorname{conv}}\, S$ and $V_{\overline{\operatorname{conv}}\, S} = \{0\}$, from which the theorem clearly follows. Thus, we assume $S^0 \neq \emptyset$.

Lemma 6 implies that $(\overline{\operatorname{conv}}\, V_{\overline{\operatorname{conv}}\, S})^0 = (V_{\overline{\operatorname{conv}}\, S})^0$ and $S^0 = (\overline{\operatorname{conv}}\, S)^0$. Proposition 8 implies $(\overline{\operatorname{conv}}\, S)^0 = (V_{\overline{\operatorname{conv}}\, S})^0$.

To show the second statement of the theorem, let C be closed and convex such that $C^0 = S^0$. Since C is closed and convex, it is enough to prove that $V_{\overline{\operatorname{conv}}(S)} \subseteq C$. Suppose, by contradiction, that this is not the case, i.e., there is an $\bar{x} \in V_{\overline{\text{conv}}(S)}$ such that $\bar{x} \notin C$. There are two cases, either $[0,1]\bar{x} \cap C = \emptyset$ or $[0,1]\bar{x} \cap C \neq \emptyset$. We will deduce a contradiction from each of them.

First, suppose $[0,1]\bar{x} \cap C = \emptyset$. Both sets are closed and $[0,1]\bar{x}$ is bounded, thus, they can be separated. Indeed, $0 \in [0,1]\bar{x}$ and [12, Corollary 11.4.1] ensure the existence of α such that $\alpha x \geq 1$ for every $x \in C$ and $\alpha\bar{x} < 1$. This means that $\alpha \in C^0$. However, $\alpha \notin (\overline{\text{conv}}\, S)^0 = S^0$, since $\bar{x} \in \overline{\text{conv}}\, S$. This contradicts $S^0 = C^0$.

Now, suppose $[0,1]\bar{x} \cap C \neq \emptyset$. Since $0, \bar{x} \notin C$, there must be $\mu \in (0,1)$ such that $\mu\bar{x} \in C$. However, $\bar{x} \in V_{\overline{\text{conv}}(S)}$ implies that $\mu\bar{x} \notin \overline{\text{conv}}(S)$. Thus, the same argument as above ensures that $\mu\bar{x}$ can be separated from $\overline{\text{conv}}(S)$. Therefore, there is an α such that $\alpha^{\mathsf{T}} x \geq 1$ for every $x \in \overline{\text{conv}}(S)$ while $\alpha^{\mathsf{T}}\mu\bar{x} < 1$. Hence, $\alpha \in S^0$ and the contradiction follows from the fact that $\mu\bar{x} \in C$ implies $\alpha \notin C^0$.

Therefore, we conclude that $\overline{\text{conv}}\, V_{\overline{\text{conv}}\, S} \subseteq C$. □

Now we show that if $\text{conv}\, S$ is closed and $0 \notin \text{conv}\, S$, then $\overline{\text{conv}}\, V_{\text{conv}\, S}$ is the answer to Question 11, i.e., is *the* smallest closed convex U such that $(U \cap S)^0 = S^0$.

Theorem 18. *Let $S \subseteq \mathbb{R}^n$ be a closed set such that* $\text{conv}\, S$ *is closed and* $0 \notin \text{conv}\, S$*, i.e., $S^0 \neq \emptyset$. Then,*

$$(\overline{\text{conv}}(V_{\text{conv}\, S}) \cap S)^0 = S^0.$$

Furthermore, if C is closed and convex such that $(C \cap S)^0 = S^0$, then

$$\overline{\text{conv}}(V_{\text{conv}\, S}) \subseteq C.$$

Proof. We first show that $(\overline{\text{conv}}(V_{\text{conv}\, S}) \cap S)^0 = S^0$.
Clearly,

$$V_{\text{conv}\, S} \cap S \subseteq \overline{\text{conv}}(V_{\text{conv}\, S}) \cap S \subseteq S.$$

Lemma 6 implies that

$$S^0 \subseteq (\overline{\text{conv}}(V_{\text{conv}\, S}) \cap S)^0 \subseteq (V_{\text{conv}\, S} \cap S)^0.$$

Thus, it is enough to show that $(V_{\text{conv}\, S} \cap S)^0 = S^0$. This follows from

$$\begin{aligned}
(S \cap V_{\text{conv}\, S})^0 &= (\text{conv}(S \cap V_{\text{conv}\, S}))^0 && \text{Lemma 6} \\
&= (\text{conv}\, V_{\text{conv}\, S})^0 && \text{Lemma 15} \\
&= (V_{\text{conv}\, S})^0 && \text{Lemma 6} \\
&= (\text{conv}\, S)^0 && \text{Proposition 8} \\
&= S^0. && \text{Lemma 6}
\end{aligned}$$

To show the second statement of the theorem, let C be a closed convex set such that $(C \cap S)^0 = S^0$. Lemma 6 implies that $(C \cap S)^0 = (\overline{\text{conv}}(C \cap S))^0$. Theorem 17 implies that $\overline{\text{conv}}(V_{\text{conv}\, S}) \subseteq \overline{\text{conv}}(C \cap S)$. Clearly, $V_{\text{conv}\, S} \subseteq \overline{\text{conv}}(V_{\text{conv}\, S})$

and $\overline{\text{conv}}(C \cap S) \subseteq C \cap \text{conv}\, S$. Therefore, $V_{\text{conv}\, S} \subseteq C \cap \text{conv}\, S$ which implies $V_{\text{conv}\, S} \subseteq C$ as we wanted. □

Finally, we answer Question 9 in the case where S is compact.

Theorem 19. *Let S be any closed set such that $0 \notin \overline{\text{conv}}\, S$. If D is any closed generator of S^0, then*

$$\overline{\text{ext}}\, V_{\overline{\text{conv}}\, S} \subseteq D.$$

If, in addition, S is compact, then $\overline{\text{ext}}\, V_{\text{conv}\, S}$ is the smallest closed generator of S^0.

Proof. First, by Lemma 6 and $D^0 = S^0$, we have $\text{shw}\, \overline{\text{conv}}\, D = \text{shw}\, \overline{\text{conv}}\, S$. Then, Lemma 7 implies that $V_{\overline{\text{conv}}\, D} = V_{\overline{\text{conv}}\, S}$. Hence, $\text{ext}\, V_{\overline{\text{conv}}\, D} = \text{ext}\, V_{\overline{\text{conv}}\, S}$. Therefore, $\text{ext}\, V_{\overline{\text{conv}}\, S} = \text{ext}\, V_{\overline{\text{conv}}\, D} \subseteq \text{ext}\, \overline{\text{conv}}\, D \subseteq D$, where the first and second containments are due to Lemma 14 and Lemma 16, respectively.

To prove the second statement, by Lemma 6, it is enough to show that $\text{ext}\, V^0_{\text{conv}\, S} = S^0$. First, as $\text{ext}\, V_{\text{conv}\, S} \subseteq \text{conv}\, S$, we have $S^0 \subset (\text{ext}\, V_{\text{conv}\, S})^0$.

To prove the other containment take any $\alpha \in (\text{ext}\, V_{\text{conv}\, S})^0$. Let $x \in \text{conv}\, S$ be arbitrary. We will prove that $\alpha^\mathsf{T} x \geq 1$. This will imply that $\alpha \in (\text{conv}\, S)^0 = S^0$ and, therefore, that $(\text{ext}\, V_{\text{conv}\, S})^0 \subseteq S^0$.
Let $\lambda \in (0, 1]$ be such that $\lambda x \in V_{\text{conv}\, S}$. If $\lambda x \in \text{ext}\, V_{\text{conv}\, S}$, then $\alpha^\mathsf{T} \lambda x \geq 1$, which implies that $\alpha^\mathsf{T} x \geq \frac{1}{\lambda} \geq 1$.
Now, assume $\lambda x \notin \text{ext}\, V_{\text{conv}\, S}$. Since S is compact, $\text{conv}\, S$ is closed and we can use Lemma 14 to obtain that $\text{ext}\, V_{\text{conv}\, S} = V_{\text{conv}\, S} \cap \text{ext}\, \text{conv}\, S$. Thus, $\lambda x \notin \text{ext}\, \text{conv}\, S$. Also by the compactness of S, [12, Theorem 18.5.1] implies that λx is a strict convex combination of some $x_1, \ldots, x_m \in \text{ext}\, \text{conv}\, S$.
Lemma 13 implies that $x_1, \ldots, x_m \in V_{\text{conv}\, S}$ and so Lemma 14 implies that $x_1, \ldots, x_m \in \text{ext}\, V_{\text{conv}\, S}$. Since $\alpha \in (\text{ext}\, V_{\text{conv}\, S})^0$, it follows $\alpha^\mathsf{T} x_i \geq 1$ for every $i = 1, \ldots, m$. Hence, $\alpha^\mathsf{T} \lambda x \geq 1$ and, as before, $\alpha^\mathsf{T} x \geq \frac{1}{\lambda} \geq 1$. □

We remark that the closure operation is needed since the extreme points of a set, in general, do not form a closed set, see [12, p. 167].

4 Applications to MINLP

Here we apply the results from Section 2 to MINLP.

In this section, unless specified otherwise, $\bar{x} \in \mathbb{R}^n$, C is a closed convex set that contains $\bar{x}$, and $S := \{x \in C : g(x) \leq 0\}$, where $g : C \to \mathbb{R}$ is continuous and $g(\bar{x}) > 0$. The idea is that C represents a convex relaxation of our MINLP and $\bar{x} \in C$ is the current relaxation solution that is infeasible for a constraint $g(x) \leq 0$.

The basic scheme for applying our results is the following translation of Observation 1.

Proposition 20. *Let $D \subseteq C$ be such that $(D \cap S)^{\bar{x}} = S^{\bar{x}}$, and $T = \{x \in D : g(x) \leq 0\}$. If $\alpha^\mathsf{T}(x - \bar{x}) \geq 1$ is a valid inequality for T, then it is valid for S.*

Proof. Directly from $\alpha \in T^{\bar{x}} = (D \cap S)^{\bar{x}} = S^{\bar{x}}$. □

Of course, the applicability of the previous proposition relies on our ability to obtain an easy-to-compute set D that satisfies the hypothesis. As shown in Section 3, $D = \overline{\text{ext}}\,\text{conv}\, V_{\text{conv}\, S}(\bar{x})$ is the smallest we can hope for, but it is useless from a practical point of view. Instead, the set of visible points of S (or a set enclosing them) is, computationally, a better candidate as we will see in Section 4.1.

Corollary 21. *Let $D \subseteq C$ be such that $V_S(\bar{x}) \subseteq D$, and $T = \{x \in D : g(x) \leq 0\}$. If $\alpha^\mathsf{T}(x - \bar{x}) \geq 1$ is a valid inequality for T, then it is valid for S.*

Proof. Clearly, $V_S(\bar{x}) \subseteq T = D \cap S \subseteq S$. The inclusion-reversing property of the reverse polar implies that $S^{\bar{x}} \subseteq (D \cap S)^{\bar{x}} \subseteq V_S(\bar{x})^{\bar{x}} = S^{\bar{x}}$, where the last equality follows from Proposition 8. The statement follows from Proposition 20. □

In the context of separation via convex underestimators Corollary 21 reads

Corollary 22. *Let $D \subseteq C$ be a closed convex set such that $V_S(\bar{x}) \subseteq D$, and let $T = \{x \in D : g(x) \leq 0\}$. If $g_D^{vex}(\bar{x}) > 0$ and $\partial g_D^{vex}(\bar{x}) \neq \emptyset$, then a gradient cut of g_D^{vex} at $\bar{x}$ is valid for S.*

Proof. Let $T_r := \{x \in D : g_D^{vex}(x) \leq 0\}$ and $v \in \partial g_D^{vex}(0)$. The cut $g_D^{vex}(0) + v^\mathsf{T} x \leq 0$ is valid for T_r, and separates 0 from T_r. Since T_r is a relaxation, i.e. $T \subseteq T_r$, it follows that the cut is also valid for T, and Corollary 21 implies its validity for S. □

The previous result tells us that if we find a box, tighter than the bounds, that contains the visible points, then we might be able to construct tighter underestimators. However, to compute a box containing $V_S(\bar{x})$ we need to know how $V_S(\bar{x})$ looks like. That is the topic of the next section.

4.1 Characterizing the visible points

From the definition of visible points we have:

Lemma 23. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $C \subseteq \mathbb{R}^n$ a closed convex set, and $S = \{x \in C : g(x) \leq 0\}$. If $\bar{x} \in C$ and $g(\bar{x}) > 0$, then*

$$V_S(\bar{x}) = \{x \in C : g(x) = 0,\ g(x + \lambda(\bar{x} - x)) > 0 \text{ for every } \lambda \in (0, 1]\}. \quad (1)$$

Furthermore, if g is differentiable, then every $x \in V_S(\bar{x})$ satisfies

$$\langle \nabla g(x), \bar{x} - x \rangle \geq 0.$$

Proof. Given that $\bar{x} \notin S$, by definition we have $x \in V_S(\bar{x})$ if and only if $x \in S$ and for every $\lambda \in (0, 1]$, $x + \lambda(\bar{x} - x) \notin C$ or $g(x + \lambda(\bar{x} - x)) > 0$. However, the convexity of C and $\bar{x} \in C$ imply that for $x \in S$, $x + \lambda(\bar{x} - x) \in C$. Hence,

$$V_S(\bar{x}) = \{x \in C \,:\, g(x) \leq 0,\ g(x + \lambda(\bar{x} - x)) > 0 \text{ for every } \lambda \in [0, 1)\}.$$

Since g is continuous, it follows that for $x \in V_S(\bar{x})$,

$$0 \geq g(x) = \lim_{\lambda \to 0^+} g(x + \lambda(x - \bar{x})) \geq 0.$$

Thus, $g(x) = 0$ which proves (1).

Now, assume that g is differentiable and let $x \in V_S(\bar{x})$. Then,

$$0 \leq \lim_{\lambda \to 0^+} \frac{g(x + \lambda(\bar{x} - x))}{\lambda} = \lim_{\lambda \to 0^+} \frac{g(x + \lambda(\bar{x} - x)) - g(x)}{\lambda} = \langle \nabla g(x), \bar{x} - x \rangle.$$

This concludes the claim. □

Remark 24. Note that if we drop the hypothesis that $\bar{x}$ is in C, then there might be visible points for which g is strictly negative, and there does not seem to be a nice description of the visible points. In such a case, $V_S(\bar{x})$ would be a disjunctive set and we would even lose the valid (non-linear) inequality $\langle \nabla g(x), \bar{x} - x \rangle \geq 0$. Likewise, if C were not convex, or if we had more than one non-convex constraint, e.g., some variable has to be binary, then there does not seem to be a nice description of the visible points. This last point is rather unfortunate, it means that it might not be easy to generalize the technique to relaxations that involve more than one non-convex constraint. In particular, since a mixed-integer set usually consists of multiple non-convex constraints, the techniques presented here might not be applicable to MILPs.
On the other hand, considering more constraints might allow us to see more of the feasible region. Therefore, in such cases one might have to try to use stronger generators such as $\overline{\operatorname{conv}}\, V_{\operatorname{conv} S}$, see also [15].

4.1.1 Quadratic constraints

For quadratic constraints, the visible points have a particularly simple description.

Theorem 25. *Let C be a closed, convex set that contains $\bar{x}$. Let $g(x) = x^\mathsf{T} Q x + b^\mathsf{T} x + c$ and $S = \{x \in C : g(x) \leq 0\}$. If $g(\bar{x}) > 0$, then*

$$V_S(\bar{x}) = \big\{x \in C \,:\, g(x) = 0,\ \langle \nabla g(\bar{x}), x \rangle + b^\mathsf{T} \bar{x} + 2c \geq 0\big\}$$

Proof. ($\subseteq$) Let $x \in V_S(\bar{x})$. By Lemma 23, we have $g(x) = 0$ and $\langle \nabla g(x), \bar{x} - x \rangle \geq 0$. Equivalently,

$$x^\mathsf{T} Q x + b^\mathsf{T} x + c = 0,$$
$$2x^\mathsf{T} Q(\bar{x} - x) + b^\mathsf{T}(\bar{x} - x) \geq 0.$$

By multiplying the equation by 2, adding it to the inequality, and re-arranging terms we obtain the result.

($\supseteq$) Let x satisfy $g(x) = 0$ and $\langle \nabla g(\bar{x}), x \rangle + b^\mathsf{T}\bar{x} + 2c \geq 0$. Then, subtracting $2g(x)$ from $\langle \nabla g(\bar{x}), x \rangle + b^\mathsf{T}\bar{x} + 2c \geq 0$ yields $\langle \nabla g(x), \bar{x} - x \rangle \geq 0$. Let

$$q(\lambda) = g(x + \lambda(\bar{x} - x)), \text{ for } \lambda \in \mathbb{R}.$$

The derivative is given by $q'(\lambda) = \langle \nabla g(x + \lambda(\bar{x} - x)), \bar{x} - x \rangle$, and $q'(0) = \langle \nabla g(x), \bar{x} - x \rangle \geq 0$. Since q is quadratic, $q(1) = g(\bar{x}) > 0$, $q(0) = g(x) = 0$, and $q'(0) \geq 0$, we have that q has no roots in $(0, 1]$. Thus, $g(x + \lambda(\bar{x} - x)) = q(\lambda) > 0$ for every $\lambda \in (0, 1]$ and, from Lemma 23, we conclude that $x \in V_S(\bar{x})$ as we wanted. □

Remark 26. Theorem 25 implies in particular that the visible points of a closed convex set intersected with a quadratic constraint, from a point in the convex set, is always closed. This does not contradict [5, Example 15.5] mentioned in Remark 12. Indeed, if one represents the cone as a quadratic constraint $q(x) \leq 0$, then the origin must be feasible for the quadratic constraint. This is easily seen from the fact that the ray $[1, \infty)(1, 0, 0)$ is in the boundary of the cone, which implies that $q(\lambda, 0, 0) = 0$ for $\lambda \geq 0$. But $q(\lambda, 0, 0)$ is a univariate quadratic function and as such can have at most two roots if it is nonzero. Hence, $q(\lambda, 0, 0) = 0$ and, in particular, $q(0, 0, 0) = 0$.

Remark 27. The hyperplane $\langle \nabla g(\bar{x}), x \rangle + b^\mathsf{T}\bar{x} + 2c = 0$ is known as the *polar hyperplane* [6] of the point $\bar{x}$ with respect to the quadratic g in projective geometry. In fact, homogenizing the quadratic g yields the quadric

$$g_h(x, x_0) = x^\mathsf{T} Q x + b^\mathsf{T} x x_0 + c x_0^2 = \begin{pmatrix} x \\ x_0 \end{pmatrix}^\mathsf{T} \begin{pmatrix} Q & \frac{b}{2} \\ \frac{b^\mathsf{T}}{2} & c \end{pmatrix} \begin{pmatrix} x \\ x_0 \end{pmatrix}.$$

The polar hyperplane of $\begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}$ with respect to $g_h(x, x_0) = 0$ is then given by

$$\langle \nabla g_h(x, x_0), (\bar{x}, 1) \rangle = 0$$
$$\iff 2\bar{x}^\mathsf{T} Q x + b^\mathsf{T}\bar{x} x_0 + b^\mathsf{T} x + 2c x_0 = 0.$$

Intersecting with $x_0 = 1$ yields $\langle \nabla g(\bar{x}), x \rangle + b^\mathsf{T}\bar{x} + 2c = 0$.

Example 5. Consider the function

$$g(x_1, x_2, x_3) = -x_1 x_2 + x_1 x_3 + x_2 x_3 - x_1 - x_2 - x_3 + 1,$$

the boxed domain $B = [-\frac{1}{10}, 2] \times [0, 2]^2$, the constrained set

$$S = \{x \in B \,:\, g(x) \leq 0\},$$

and the infeasible point $\bar{x} = (0, 0, 0)$. By Theorem 25, the visible points from $\bar{x}$ are given by

$$V_S(\bar{x}) = \{(x_1, x_2, x_3) \in B : g(x) = 0,\ x_1 + x_2 + x_3 \geq 0\},$$

as shown in Figure 3.

The tightest box bounding V_S is

$$R = [-\frac{1}{10}, 1] \times [0, \frac{1}{20}(23 + 3\sqrt{5})] \times [0, \frac{1}{20}(19 + 3\sqrt{5})].$$

The linear underestimators of g obtained by using McCormick [9] inequalities for each term over B and R are

$$1 \leq x_1 + 3x_2 + \frac{11}{10}x_3 \text{ and } 1 \leq x_1 + 2x_2 + \frac{11}{10}x_3,$$

respectively. Since $0 \leq x_2$, it follows that the underestimator over R dominates the underestimator over B. We remark that the improvement in this particular cut is only due to the improvement on the upper bound of x_1.

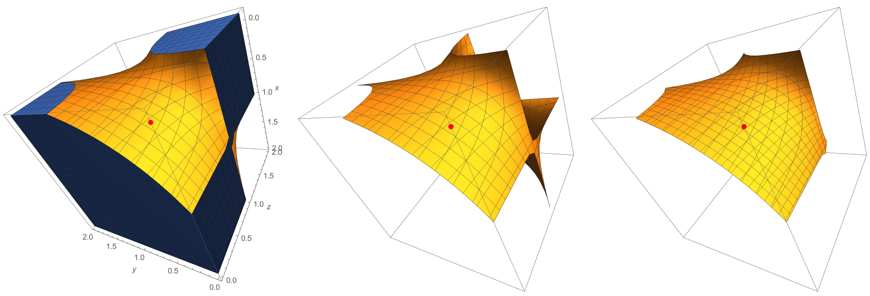

Figure 3: The left plot shows the feasible region S and $\bar{x}$. The set $\{x \in B : g(x) = 0\}$ appears in the middle plot. Finally, the visible points, $V_S(\bar{x})$, are plotted on the right.

□

4.1.2 Polynomial constraints

For a general polynomial g, the condition

$$g(x + \lambda(\bar{x} - x)) > 0 \text{ for every } \lambda \in (0, 1] \tag{2}$$

of (1) asks for the univariate polynomial $p_x(\lambda) = g(x + \lambda(\bar{x} - x))$ to be positive on $(0, 1]$. We can then use the theory of non-negative polynomials to translate a relaxation of the infinitely many constraints (2) to a finite number of constraints. From the following characterization of non-negative polynomials on intervals we can derive an extended formulation for the relaxation of (1),

$$R_S(\bar{x}) := \{x \in C : g(x) = 0,\ g(x + \lambda(\bar{x} - x)) \geq 0 \text{ for every } \lambda \in [0, 1]\}.$$

Theorem 28. *Let $p \in \mathbb{R}[\lambda]$ be a polynomial. Then p is non-negative on $[0, 1]$ if and only if*

1. *the degree of p is $2d$ and there exist $s_1, s_2 \in \mathbb{R}[\lambda]$ of degree d and $d-1$, respectively, such that*

$$p(\lambda) = s_1(\lambda)^2 + \lambda(1-\lambda)s_2(\lambda)^2.$$

2. *the degree of p is $2d+1$ and there exist $s_1, s_2 \in \mathbb{R}[\lambda]$ of degree d, such that*

$$p(\lambda) = \lambda s_1(\lambda)^2 + (1-\lambda)s_2(\lambda)^2.$$

Proof. See [10]. □

Theorem 29. *Let C be a closed convex set that contains $\bar{x}$. Let $g(x)$ be a polynomial such that $g(\bar{x}) > 0$ and $S = \{x \in C : g(x) \leq 0\}$. Let $p_x(\lambda) = g(x + \lambda(\bar{x} - x))$.*

1. *If the degree of g is $2d$, then*

$$R_S(\bar{x}) = \operatorname{proj}_x E,$$

where E is

$$\{(x, A, B) \in C \times \mathcal{S}_+^d \times \mathcal{S}_+^d : \\ g(x) = 0, \\ p_x'(0) = B_{00}, \\ \frac{p_x^{(k+2)}(0)}{(k+2)!} = \sum_{\substack{i+j=k \\ 0 \leq i,j \leq d-1}} A_{ij} - B_{ij} + \sum_{\substack{i+j=k+1 \\ 0 \leq i,j \leq d-1}} B_{ij}, \text{ for } 0 \leq k \leq 2d-2\}.$$

2. *If the degree of g is $2d+1$, then*

$$R_S(\bar{x}) = \operatorname{proj}_x E,$$

where E is

$$\{(x, A, B) \in C \times \mathcal{S}_+^{d+1} \times \mathcal{S}_+^d : \\ g(x) = 0, \\ p_x'(0) = A_{00}, \\ \frac{p_x''(0)}{2} = 2A_{01} + B_{00}, \\ \frac{p_x^{(k+3)}(0)}{(k+3)!} = \sum_{\substack{i+j=k+2 \\ 0 \leq i,j \leq d}} A_{ij} + \sum_{\substack{i+j=k+1 \\ 0 \leq i,j \leq d-1}} B_{ij} - \sum_{\substack{i+j=k \\ 0 \leq i,j \leq d-1}} B_{ij}, \text{ for } 0 \leq k \leq 2d-2\}.$$

Proof. We just prove the case of even degree as the proof for the odd degree case is similar. We have $x \in R_S(\bar{x})$ if and only if $p_x(0) = 0$ and $p_x(\lambda)$ is non-negative on $[0, 1]$. By Theorem 28, this is equivalent to $p_x(0) = 0$ and there exist polynomials s_1, s_2 of degree d and $d-1$, respectively, such that

$$p_x(\lambda) = s_1(\lambda)^2 + \lambda(1-\lambda)s_2(\lambda)^2.$$

Given that $0 = p_x(0) = s_1(0)^2$, the polynomial s_1 has a root at 0 and we can write it as $s_1(\lambda) = \lambda r_1(\lambda)$ where r_1 is a polynomial of degree $d-1$. Thus, $x \in R_S(\bar{x})$ if and only if $p_x(0) = 0$ and there exist polynomials r_1, r_2 of degree $d-1$ such that

$$p_x(\lambda) = \lambda^2 r_1(\lambda)^2 + \lambda(1-\lambda) r_2(\lambda)^2.$$

Let $\Lambda = (1, \lambda, \ldots, \lambda^{d-1})^\mathsf{T}$. The polynomials r_i can be written as $r_i = c_i^\mathsf{T}\Lambda$ for some $c_i \in \mathbb{R}^d$. Then, $r_1(\lambda)^2 = \Lambda^\mathsf{T} A \Lambda$ and $r_2(\lambda)^2 = \Lambda^\mathsf{T} B \Lambda$ for some $A, B \in \mathcal{S}_+^d$.

Thus, $x \in R_S(\bar{x})$ if and only if $p_x(0) = 0$ and there exist $A, B \in \mathcal{S}_+^d$ such that

$$p_x(\lambda) = \lambda^2 \Lambda^\mathsf{T} A \Lambda + \lambda(1-\lambda)\Lambda^\mathsf{T} B \Lambda.$$

Since $p_x(\lambda)$ is a polynomial of degree $2d$, its Taylor expansion at 0 yields

$$p_x(\lambda) = \sum_{k=1}^{2d} \frac{p_x^{(k)}(0)}{k!}\lambda^k.$$

Identifying coefficients, we conclude the theorem. □

Remark 30. One could also add the constraints $\mathrm{rk}(A) = \mathrm{rk}(B) = 1$ to E in the statement of Theorem 29. The correctness can easily be seen from the proof since $A = c_1 c_1^\mathsf{T}$ and $B = c_2 c_2^\mathsf{T}$. Although it makes the set more restricted, the rank constraint is non-convex and does not change the projection. Thus, we decided to leave it out.

We can recover Theorem 25 from Theorem 29. The set E of Theorem 29 for the quadratic case ($d = 1$) is described by $g(x) = 0$, $p'_x(0) = B_{00}$ and $p''_x(0)/2 = A_{00} - B_{00}$, where $A_{00}, B_{00} \geq 0$. This implies that $0 < g(\bar{x}) = p_x(1) = p'_x(0) + p''_x(0)/2 = A_{00}$. Therefore, $R_S(\bar{x})$ consists of the x such that $p_x(0) = 0$ and $p'_x(0) \geq 0$. This last constraint is equivalent to $\langle \nabla g(x), \bar{x} - x \rangle \geq 0$ which is the only constraint needed, apart from $g(x) = 0$, to prove Theorem 25.

The previous deduction is only possible because $V_S(\bar{x}) = R_S(\bar{x})$ holds for a quadratic constraint. This equality does not hold as soon as the degree is greater than 2, even after replacing $V_S(\bar{x})$ by its closure, as shown in the following example.

Example 6. Consider $g(x_1, x_2) = (x_1^2 + x_2^2 - 1)x_1$, $S = \{(x_1, x_2) : g(x_1, x_2) \leq 0\}$, and $\bar{x} = (1, -2)$. The set S consists of the right half of the unit ball and the half space $x_1 \leq 0$ without the interior of the left half of the unit ball, see Figure 4. The point $z = (-1, 0)$ is not visible from $\bar{x}$, because $g(z + \lambda(\bar{x} - z)) = g(-1 + 2\lambda, -2\lambda) = ((2\lambda - 1)^2 + 4\lambda^2 - 1)(2\lambda - 1) = 4\lambda(2\lambda - 1)^2$ is zero at $\lambda = \frac{1}{2}$. On the other hand, $z \in R_S(\bar{x})$ since $4\lambda(2\lambda - 1)^2 \geq 0$ for every $\lambda \in [0, 1]$. In this example $V_S(\bar{x})$ is closed, so we conclude that $\operatorname{cl} V_S(\bar{x}) \neq R_S(\bar{x})$. □

5 Conclusions and outlook

Using the concept of visible points, we introduced a technique that allows to reduce the domains in separation problems. Such a result is particularly inter-

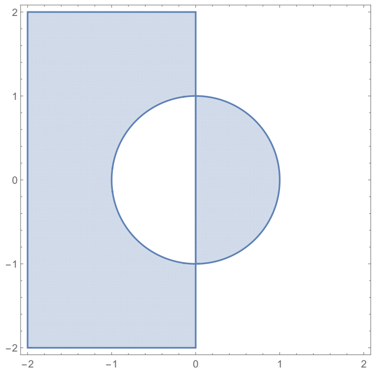

Figure 4: Feasible region $g(x) \leq 0$ of Example 6 that shows that $\operatorname{cl} V_S(\bar{x}) \neq R_S(\bar{x})$ when the degree of g is greater than 2.

esting for MINLP, since the tightness of the domain directly affects the quality of underestimators, from which cuts are obtained.

Some questions that could be interesting to look at in the future are the following. Is there a tighter domain other than V_S that can be efficiently exploited? Is there a useful characterizations of V_S when S contains more than one non-convex constraint, in particular, if some variables are restricted to be integer?

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