KAI HOPPMANN (iD), FELIX HENNINGS (iD), RALF LENZ (iD), THORSTEN KOCH (iD)

Optimal Operation of Macroscopic Gas Transport Networks Over Time

Optimal Operation of Macroscopic Gas Transport Networks Over Time

Kai Hoppmann

June 28, 2019

Abstract

In this report we introduce a hierarchical MIP formulation for an optimal macroscopic operation of transient gas transport networks. We call it macroscopic operation, since we use a simplified model for the highly complex intersection areas of the gas network, which we call navi stations in the following. Their capabilities are modelled using artificial links, which can in- or decrease the gas pressure or act as short connecting pipes. Deciding on a predefined flow direction for each station, which describes where gas enters and leaves it, for each step in time we additionally have to choose a so-called simple state, which corresponds to a subset of the artificial links that can be used for gas transport. Further, in order to derive a linear model for the transient gas flow between navi stations, we linearize the Euler equations describing the gas flow in pipes. In particular, we simplify the pressure loss due to friction by fixing the absolute velocity in the corresponding equation. In order to make up for this crucial simplification, an iterative adjustment procedure follows as postprocessing with a subsequent solution smoothening. The goal of our optimization problem is to ensure that all demands are satisfied, while the usage of technical and non-technical measures is minimized, i.e., by minimizing the change of simple states over time and the manipulation of supplies and demands via slack variables.

1 General Concepts and Definitions

1.1 Network Topology

The Netmodel is based on a directed graph $G = (\mathcal{V}, \mathcal{A}^{\mathrm{pi}} \cup \mathcal{A}^{\mathrm{rg}} \cup \mathcal{A}^{\mathrm{va}} \cup \mathcal{A}^{\mathrm{ar}})$, where $\mathcal{V}$ denotes the set of *nodes*, $\mathcal{A}^{\mathrm{pi}}$ the set of *pipes*, $\mathcal{A}^{\mathrm{rg}}$ the set of *regulators*, $\mathcal{A}^{\mathrm{va}}$ the set of *valves*, and $\mathcal{A}^{\mathrm{ar}}$ the set of so-called *artificial links*. We allow parallel and antiparallel arcs, but we do not allow loops. Furthermore, there are two sets $\mathcal{V}^+ \subseteq \mathcal{V}$ and $\mathcal{V}^- \subseteq \mathcal{V}$ called *entries* and *exits* of the network, respectively. Additionally, $\mathcal{V}^{\mathrm{b}} := \mathcal{V}^+ \cup \mathcal{V}^-$ is called the set of *boundary nodes*, while $\mathcal{V}^0 := \mathcal{V} \backslash \mathcal{V}^{\mathrm{b}}$ is called the set of *inner nodes*.

1.2 Navi Stations

Within G there exist $m \in \mathbb{N}$ subgraphs $G_i = (\mathcal{V}_i, \mathcal{A}_i^{\mathrm{ar}})$ with $\mathcal{V}_i \subseteq \mathcal{V}^0$ and $\mathcal{A}_i^{\mathrm{ar}} \subseteq \mathcal{A}^{\mathrm{ar}}$ for each $i \in \{1, \ldots, m\}$, which are called *navi stations*. It holds that $\bigcup_{i=1}^m \mathcal{A}_i^{\mathrm{ar}} = \mathcal{A}^{\mathrm{ar}}$. Further, each artificial link is contained in exactly one navi

station and navi stations do not share common nodes, i.e., $\mathcal{A}_i^{\mathrm{ar}} \cap \mathcal{A}_j^{\mathrm{ar}} = \emptyset$ and $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ for $i,j \in \{1,\dots,m\}$ with $i \neq j$.

1.3 Remaining Network

The remaining network $(\tilde{\mathcal{V}}, \tilde{\mathcal{A}} := \mathcal{A}^{\mathrm{pi}} \cup \mathcal{A}^{\mathrm{rg}} \cup \mathcal{A}^{\mathrm{va}})$ consists of pipes, regulators, and valves, together with the set of all nodes that are incident to at least one of the former elements, i.e., $\tilde{\mathcal{V}} := \{v \in \mathcal{V} \,|\, \delta(v) \cap \tilde{\mathcal{A}} \neq \emptyset\} \subseteq \mathcal{V}$. The nodes $\mathcal{V}^{\mathrm{fn}} := \tilde{\mathcal{V}} \cap \bigcup_{i=1}^m \mathcal{V}_i$ connecting the navi stations with the remaining network are called *fence nodes* while $\mathcal{V}^{\mathrm{ar}} := \mathcal{V} \setminus \tilde{\mathcal{V}}$ are called *artificial nodes*. Hence, a navi station consists of fence nodes $\mathcal{V}_i^{\mathrm{fn}}$ and artificial nodes $\mathcal{V}_i^{\mathrm{ar}}$, i.e., we have $\mathcal{V}_i = \mathcal{V}_i^{\mathrm{fn}} \dot{\cup} \mathcal{V}_i^{\mathrm{ar}}$.

1.4 Timesteps and Granularity

For the Netmodel we are given a set of *timesteps* $\mathcal{T}_0 := \{0,\dots,k\}$ together with a monotonically increasing function $\tau : \mathcal{T}_0 \to \mathbb{N}$, i.e., $\tau(t-1) < \tau(t)$ for all $t \in \mathcal{T}_0 \setminus \{0\}$, with $\tau(0) = 0$. Here, $\tau(t)$ describes the number of seconds that pass until time step $t \in \mathcal{T}_0$ w.r.t. 0. For notational purposes we define $\mathcal{T} := \mathcal{T}_0 \setminus \{0\}$ as the set of timesteps without zero.

2 Node-Related Variables and Constraints

2.1 Supply and Demand Variables

For each each entry $u \in \mathcal{V}^+$ and each $t \in \mathcal{T}$ we are given a *supply* value $D_{u,t} \in \mathbb{R}_{\geq 0}$ and for each exit $w \in \mathcal{V}^-$ and each $t \in \mathcal{T}$ we are given a *demand* value $D_{w,t} \in \mathbb{R}_{\leq 0}$. These supplies and demands can be adjusted using slack variables. Thus, for each boundary node $v \in \mathcal{V}^{\mathrm{b}}$ and each $t \in \mathcal{T}$ we introduce two continuous variables $\sigma_{v,t}^{d+}, \sigma_{v,t}^{d-} \in \mathbb{R}_{\geq 0}$. The actual supplies considered in the model are then given by variables $d_{u,t} \in [0, \bar{D}_{u,t}]$ and the actual demands by $d_{w,t} \in [\underline{D}_{w,t}, 0]$. Here, $\overline{D}_u$ is an upper bound on u's supply and $\underline{D}_w$ is a lower bound on w's demand. Finally, we introduce the following constraints:

$$d_{u,t} + \sigma_{u,t}^{d+} - \sigma_{u,t}^{d-} = D_{u,t} \quad \forall u \in \mathcal{V}^+, \forall t \in \mathcal{T} \text{ and} \tag{1}$$

$$d_{w,t} + \sigma_{w,t}^{d+} - \sigma_{w,t}^{d-} = D_{w,t} \quad \forall w \in \mathcal{V}^-, \forall t \in \mathcal{T}. \tag{2}$$

The slack variables $\sigma_{v,t}^{d+}$ and $\sigma_{v,t}^{d-}$ contribute to the objective function with cost $w^{\sigma\text{-d}} \in \mathbb{R}_{\geq 0}$.

2.2 Initial Pressure and Pressure Variables

For each node $v \in \mathcal{V}$ we are given an initial nonnegative pressure value $p_{v,0} \in \mathbb{R}_{\geq 0}$. Further, for each node $v \in \mathcal{V}$ and each point in time $t \in \mathcal{T}$ we introduce a nonnegative pressure variable $p_{v,t} \in [\underline{p}_{v,t}, \bar{p}_{v,t}] \subseteq \mathbb{R}_{\geq 0}$, where $\underline{p}_{v,t}$ is a lower and $\bar{p}_{v,t}$ an upper bound on the pressure at node v and time t. These bounds we call *technical pressure bounds*.

2.3 Inflow Pressure Bounds and Slacks

For each boundary node $v \in \mathcal{V}^{\mathrm{b}}$ and each point in time $t \in \mathcal{T}$ we are given so-called *inflow pressure bounds*. These pressure bounds are in general tighter than the technical pressure bounds and have to be respected if a boundary node has nonzero supply or demand. We denote them by $\underline{p}_{v,t}^{\mathrm{act}}$ and $\bar{p}_{v,t}^{\mathrm{act}}$.

These bounds may be be relaxed using slack to ensure feasibility and therefore we introduce two variables $\sigma_{v,t}^{p-} \in [0, \bar{p}_{v,t} - \bar{p}_{v,t}^{\mathrm{act}}]$ and $\sigma_{v,t}^{p+} \in [0, \underline{p}_{v,t}^{\mathrm{act}} - \underline{p}_{v,t}]$ and the following two types of constraints:

$$p_{v,t} - \sigma_{v,t}^{p-} \leq \bar{p}_{v,t}^{\mathrm{act}} \quad \forall v \in \mathcal{V}^{\mathrm{b}} \text{ with } D_{v,t} \neq 0, \forall t \in \mathcal{T} \text{ and} \tag{3}$$

$$p_{v,t} + \sigma_{v,t}^{p+} \geq \underline{p}_{v,t}^{\mathrm{act}} \quad \forall v \in \mathcal{V}^{\mathrm{b}} \text{ with } D_{v,t} \neq 0, \forall t \in \mathcal{T}. \tag{4}$$

The variables $\sigma_{v,t}^{p-}$ and $\sigma_{v,t}^{p+}$ contribute to the objective function with cost $w^{\sigma\text{-p}} \in \mathbb{R}_{\geq 0}$.

Note: We currently do not enforce the inflow pressure bounds dynamically. This means that if, for example, $D_{u,t} = 0$ and $d_{u,t} > 0$ for some source $u \in \mathcal{V}^+$ and some point in time $t \in \mathcal{T}$, then the inflow pressure bounds are not enforced. Conversely, if, for example, $D_{w,t} < 0$ but $d_{w,t} = 0$ for some sink $w \in \mathcal{V}^-$ and some point in time $t \in \mathcal{T}$, then the inflow pressure bound constraints (3) and (4) still have to be satisfied. To model a dynamical behaviour, we would need to introduce a new class of binary variables, which would serve as indicator whether or not there is in- or outflow at a boundary node, together with the corresponding constraints.

3 Elements in the Remaining Network

3.1 Pipes

Let $a = (\ell, r) \in \mathcal{A}^{\mathrm{pi}}$ be a pipe in G. The flow on it is described by two types of constraints, which we introduce for each timestep $t \in \mathcal{T}$. They are derived from the continuity equation (5), which captures the transient behaviour of the gas flow, and the momentum equation (6), which determines the pressure loss, of the Euler equations.

$$p_{\ell,t} - p_{\ell,t-1} + p_{r,t} - p_{r,t-1} + \frac{2\,R_s\,\Delta t}{L_a\,A_a}\left(T_r\,z_r\,q_{r,a,t} - T_\ell\,z_\ell\,q_{\ell,a,t}\right) = 0 \tag{5}$$

$$p_{r,t} - p_{\ell,t} + \frac{\lambda_a\,L_a}{4\,A_a\,D_a}\left(|v_{\ell,0}|\,q_{\ell,a,t} + |v_{r,0}|\,q_{r,a,t}\right) + \frac{g\,s_a\,L_a}{2\,R_s}\left(T_\ell\,z_\ell\,p_{\ell,t} + T_r z_r p_{r,t}\right) = 0 \tag{6}$$

The variable $q_{\ell,a,t} \in [\underline{q}_{a,t}, \bar{q}_{a,t}] \subseteq \mathbb{R}$ describes the flow into the pipe at node ℓ while $q_{r,a,t} \in [\underline{q}_{a,t}, \bar{q}_{a,t}] \subseteq \mathbb{R}$ is the flow out of the pipe at node r. The parameters $\underline{q}_{a,t}, \bar{q}_{a,t} \in \mathbb{R}$ denote technical lower and upper bounds on the flow. Note that both variables may take on negative values meaning that flow is going out of the pipe at node ℓ or into the pipe at node r, respectively. The parameters used in the constraints are (equivalently for node r if applicable):

- $R_s :=$ specific gas constant

3

- $\Delta t = \tau(t) - \tau(t-1) :=$ length of considered time interval in s

- $L_a :=$ length of the pipe in m

- $A_a :=$ area of the pipe in m^2

- $T_\ell :=$ gas temperature in K at ℓ (constant here - depends on pressure)

- $z_\ell :=$ compressibility factor at ℓ (constant here - depends on pressure)

- $\lambda_a :=$ friction factor of the pipe (constant here - depends on flow)

- $D_a :=$ diameter of the pipe in m

- $|v_{\ell,0}| :=$ absolute velocity at ℓ and $t = 0$ (constant here - depends on flow and pressure)

- $g :=$ gravitational acceleration $9.81\frac{m}{s^2}$

- $s_a = \frac{h_r - h_\ell}{L_a} :=$ slope of pipe a with h_ℓ being the altitude at ℓ

Remark: An important simplification we are making is that we fix the absolute velocity in the friction term of the momentum equation (6) to the absolute velocity at the corresponding node at time 0. To overcome this simplification, the iterative algorithmic procedure in Section 6.4 is introduced.

3.2 Regulators

Let $a = (\ell, r) \in \mathcal{A}^{\mathrm{rg}}$ be a regulator in G. For each step in time $t \in \mathcal{T}$ we have a flow variable $q_{a,t} \in [0, \bar{q}_{a,t}] \subseteq \mathbb{R}_{\geq 0}$, i.e., only flow from ℓ towards r is possible. Further, the pressure at ℓ is not allowed to be smaller than the pressure at r:

$$p_{\ell,t} - p_{r,t} \geq 0 \quad \forall t \in \mathcal{T}. \tag{7}$$

Note: In order to model flap traps, i.e., that the pressure at r can be larger than at ℓ if there is no flow, we would need to introduce an additional class of binary variables, indicating whether or not there is flow on the regulator. If there is flow, (7) would be active, and if not, the pressures at ℓ and r would be decoupled. But since all regulators in the network are bordering elements to distribution networks or are located directly behind an entry, there is no need to capture this feature so far.

3.3 Valves

Let $a = (\ell, r) \in \mathcal{A}^{\mathrm{va}}$ be a valve in G. For each step in time $t \in \mathcal{T}$ we have a forward and a backward flow variable $\overrightarrow{q}_{a,t}, \overleftarrow{q}_{a,t} \in [0, \bar{q}_{a,t}] \subseteq \mathbb{R}_{\geq 0}$. Further, we introduce a binary variable $z_{a,t} \in \{0,1\}$ indicating whether the valve is open or not. If the valve is open, the pressure have to be equal and there can be flow in an arbitrary direction. If it is closed, there is no flow and the pressures are decoupled.

$$p_{\ell,t} - p_{r,t} \leq (1 - z_{a,t})(\bar{p}_{\ell,t} - \underline{p}_{r,t}) \quad \forall a = (\ell, r) \in \mathcal{A}^{\mathrm{va}}, \ \forall t \in \mathcal{T} \tag{8}$$

$$p_{\ell,t} - p_{r,t} \geq (1 - z_{a,t})(\underline{p}_{\ell,t} - \bar{p}_{r,t}) \quad \forall a = (\ell, r) \in \mathcal{A}^{\mathrm{va}}, \ \forall t \in \mathcal{T} \tag{9}$$

$$\overrightarrow{q}_{a,t} \leq \bar{q}_{a,t} z_{a,t} \qquad \qquad \forall a \in \mathcal{A}^{\mathrm{va}}, \ \forall t \in \mathcal{T} \tag{10}$$

$$\overleftarrow{q}_{a,t} \leq \bar{q}_{a,t} z_{a,t} \qquad \qquad \forall a \in \mathcal{A}^{\mathrm{va}}, \ \forall t \in \mathcal{T} \tag{11}$$

4 Navi Stations

Recall, that within G there are subgraphs $G_i = (\mathcal{V}_i, \mathcal{A}_i^{\mathrm{ar}})$ with $i \in \{1, \ldots, m\}$ called *navi stations*, see Section 1.2. We demonstrate the following definitions all with an example.

$\mathcal{V}_i = \mathcal{V}_i^{\mathrm{fn}} \cup \mathcal{V}_i^{\mathrm{ar}} \subseteq \mathcal{V}^0$ are the fence and artificial nodes the station consists of.

- Example: $\mathcal{V}_i = \{\mathrm{nord}, \mathrm{sued}, \mathrm{remich}, \mathrm{gerns}, \mathrm{medel}, \mathrm{creos}, \mathrm{t}, \mathrm{m}\}$ with $\mathcal{V}_i^{\mathrm{fn}} = \{\mathrm{nord}, \mathrm{sued}, \mathrm{remich}, \mathrm{gerns}, \mathrm{medel}, \mathrm{creos}\}$ and $\mathcal{V}_i^{\mathrm{ar}} = \{\mathrm{t}, \mathrm{m}\}$

$\mathcal{A}_i^{\mathrm{ar}} \subseteq \mathcal{V}_i \times \mathcal{V}_i \subseteq \mathcal{A}^{\mathrm{ar}}$ is the set of artificial links of the navi station. There are seven types of artificial links:

- $\mathcal{A}_i^{\mathrm{ar\text{-}sc\text{-}bi}} \subseteq \mathcal{A}_i^{\mathrm{ar}}$ the set of *(bidirected) shortcuts* of the navi station.

- $\mathcal{A}_i^{\mathrm{ar\text{-}rg}} \subseteq \mathcal{A}_i^{\mathrm{ar}}$ the set of *(directed) regulating links* of the navi station.

- $\mathcal{A}_i^{\mathrm{ar\text{-}rg\text{-}bi}} \subseteq \mathcal{A}_i^{\mathrm{ar}}$ the set of *bidirected regulating links* of the navi station.

- $\mathcal{A}_i^{\mathrm{ar\text{-}cp}} \subseteq \mathcal{A}_i^{\mathrm{ar}}$ the set of *(directed) compressing links* of the navi station.

- $\mathcal{A}_i^{\mathrm{ar\text{-}cp\text{-}bi}} \subseteq \mathcal{A}_i^{\mathrm{ar}}$ the set of *bidirected compressing links* of the navi station.

- $\mathcal{A}_i^{\mathrm{ar\text{-}cb}} \subseteq \mathcal{A}_i^{\mathrm{ar}}$ the set of *(directed) combined links* of the navi station.

- $\mathcal{A}_i^{\mathrm{ar\text{-}cb\text{-}bi}} \subseteq \mathcal{A}_i^{\mathrm{ar}}$ the set of *bidirected combined links* of the navi station.

Example:

- $\mathcal{A}_i^{\mathrm{ar\text{-}sc\text{-}bi}} = \{n := (\mathrm{nord}, \mathrm{sued}), \mathit{hinter} := (\mathrm{t}, \mathrm{medel}), \mathit{vor} := (\mathrm{m}, \mathrm{t}), gm := (\mathrm{gerns}, \mathrm{m}), bp := (\mathrm{gerns}, \mathrm{medel}), cs := (\mathrm{creos}, \mathrm{sued})\}$

- $\mathcal{A}_i^{\mathrm{ar\text{-}rg}} = \{\mathit{rem} := (\mathrm{gerns}, \mathrm{remich})\}$

- $\mathcal{A}_i^{\mathrm{ar\text{-}rg\text{-}bi}} = \{ms := (\mathrm{t}, \mathrm{sued}), mn := (\mathrm{t}, \mathrm{nord})\}$

- $\mathcal{A}_i^{\mathrm{ar\text{-}cp}} = \{vst := (\mathrm{nord}, \mathrm{sued}), gs := (\mathrm{t}, \mathrm{sued}), vsm := (\mathrm{m}, \mathrm{medel})\}$

- $\mathcal{A}_i^{\mathrm{ar\text{-}cp\text{-}bi}} = \mathcal{A}_i^{\mathrm{ar\text{-}cb}} = \mathcal{A}_i^{\mathrm{ar\text{-}cb\text{-}bi}} = \emptyset$

By $\mathcal{A}_i^{\mathrm{ar\text{-}di}} := \mathcal{A}_i^{\mathrm{ar\text{-}rg}} \cup \mathcal{A}_i^{\mathrm{ar\text{-}cp}} \cup \mathcal{A}_i^{\mathrm{ar\text{-}cb}}$ we denote the set of all directed artificial arcs of navi station G_i, where only flow into the forward direction possible. By $\mathcal{A}_i^{\mathrm{ar\text{-}bi}} := \mathcal{A}_i^{\mathrm{ar\text{-}sc\text{-}bi}} \cup \mathcal{A}_i^{\mathrm{ar\text{-}rg\text{-}bi}} \cup \mathcal{A}_i^{\mathrm{ar\text{-}cp\text{-}bi}} \cup \mathcal{A}_i^{\mathrm{ar\text{-}cb\text{-}bi}}$ we denote the set of all bidirected artificial arcs, where flow into both directions, forward and backward, is possible. Furthermore, by $\mathcal{A}_i^{\mathrm{ar\text{-}co}} := \mathcal{A}_i^{\mathrm{ar\text{-}cp}} \cup \mathcal{A}_i^{\mathrm{ar\text{-}cp\text{-}bi}} \cup \mathcal{A}_i^{\mathrm{ar\text{-}cb}} \cup \mathcal{A}_i^{\mathrm{ar\text{-}cb\text{-}bi}}$ we denote the set of links, that are able to compress gas.

$\mathcal{F}_i \subseteq \mathcal{P}(\mathcal{V}_i) \times \mathcal{P}(\mathcal{V}_i)$, where $\mathcal{P}$ denotes the powerset, is called the set of flow directions of navi station G_i. A flow direction $f = (f^+, f^-) \in \mathcal{F}_i$ consists of entry fence nodes f^+ and exit fence nodes f^- and we assume that $f^+ \cap f^- = \emptyset$. Example:

- $\mathrm{ng} := (\{\mathrm{nord}, \mathrm{gerns}\}, \{\mathrm{sued}, \mathrm{medel}, \mathrm{remich}, \mathrm{creos}\})$

- `n-g` := ({nord}, {gerns, sued, medel, remich, creos})

- `g-n` := ({gerns}, {nord, sued, medel, remich, creos})

Let $\mathbb{N}_{\mathcal{S}} := \{0, \ldots, 400\}$. The set $\mathcal{S}_i \subseteq \mathbb{N}_{\mathcal{S}} \times \mathcal{P}(\mathcal{F}_i) \times \mathcal{P}(\mathcal{A}_i^{\mathrm{ar}}) \times \mathcal{P}(\mathcal{A}_i^{\mathrm{ar}})$ is called the set of *simple states* of navi station G_i. A simple state $s = (w^s, s^f, s^{on}, s^{off}) \in \mathcal{S}_i$ consists of its initialization cost w^s, its supported flow directions s^f, its set of active artificial links s^{on}, and its set of inactive artificial links s^{off}. Note, that $s^f \neq \emptyset$. Example:

- `TB-MB` := $(320, \{ng\}, \{n, cs, bp, rem\}, \{vsm, gm, vor, hinter, vst, gs, ms, mn\})$

- `TVvo-MBvo` := $(145, \{ng\}, \{vst, cs, mn, vor, gm, bp, rem\}, \{vsm, hinter, n, gs, ms\})$

- `TBhi-MBvo` := $(5, \{ng, \ g-n\}, \{n, cs, ms, vor, gm, bp, rem\}, \{vsm, hinter, vst, gs, mn\})$

4.1 Configuration Choosing

Next, for each station, at each point in time exactly one flow direction $f \in \mathcal{F}_i$ has to be active. Given such a flow direction, one must choose exactly one simple state $s \in \mathcal{S}_i$ for which $f \in s^f$ (except for $t = 0$, where the flow direction and the simple state are decoupled). Given a decision on the simple state, all the active links of the simple state s^{on} must be active, while all the inactive links s^{off} must be inactive. All remaining (optional) artificial links $a \in \mathcal{A}_i^{\mathrm{ar}} \setminus (s^{on} \cup s^{off})$ may be active, but do not have to. We model these requirements using the following variables

- $x_{f,t} \in \{0, 1\}$ if flow direction $f \in \mathcal{F}_i$ is active at time $t \in \mathcal{T}_0$

- $x_{s,t} \in \{0, 1\}$ if simple state $s \in \mathcal{S}_i$ is active at time $t \in \mathcal{T}_0$

- $x_{a,t} \in \{0, 1\}$ if artificial link $a \in \mathcal{A}_i^{\mathrm{ar}}$ is active at time $t \in \mathcal{T}_0$

and the following constraints

$$\sum_{f \in \mathcal{F}_i} x_{f,t} = 1 \qquad \forall t \in \mathcal{T}_0 \tag{12}$$

$$\sum_{f \in s^f} x_{f,t} \geq x_{s,t} \quad \forall s \in \mathcal{S}_i, \forall t \in \mathcal{T} \tag{13}$$

$$\sum_{s \in \mathcal{S}_i} x_{s,t} = 1 \qquad \forall t \in \mathcal{T}_0 \tag{14}$$

$$x_{s,t} \leq x_{a,t} \quad \forall s \in \mathcal{S}_i, \forall a \in s^{on}, \forall t \in \mathcal{T}_0 \tag{15}$$

$$1 - x_{s,t} \geq x_{a,t} \quad \forall s \in \mathcal{S}_i, \forall a \in s^{off}, \forall t \in \mathcal{T}_0. \tag{16}$$

In order to account for changes in the flow direction, simple state, and artificial links over time, we introduce the following variables

- $\delta_{f,t} \in \{0, 1\}$ for each $f \in \mathcal{F}_i$ and each $t \in \mathcal{T}$

- $\delta_{s,t} \in \{0, 1\}$ for each $s \in \mathcal{S}_i$ and each $t \in \mathcal{T}$

- $\delta_{a,t}^{on}, \delta_{a,t}^{off} \in \{0, 1\}$ for each $a \in \mathcal{A}_i^{\mathrm{ar}}$ and each $t \in \mathcal{T}$

and add the following constraints

$$x_{f,t-1} - x_{f,t} + \delta_{f,t} \geq 0 \quad \forall f \in \mathcal{F}_i, \forall t \in \mathcal{T} \tag{17}$$

$$x_{s,t-1} - x_{s,t} + \delta_{s,t} \geq 0 \quad \forall s \in \mathcal{S}_i, \forall t \in \mathcal{T} \tag{18}$$

$$x_{a,t-1} - x_{a,t} + \delta_{a,t}^{\mathrm{on}} - \delta_{a,t}^{\mathrm{off}} = 0 \quad \forall a \in \mathcal{A}_i^{\mathrm{ar}}, \forall t \in \mathcal{T} \tag{19}$$

While $\delta_{f,t}, \delta_{s,t}, \delta_{a,t}^{\mathrm{on}}$ indicate whether or not a flow direction, simple state, or artificial link has been switched on, $\delta_{a,t}^{\mathrm{off}}$ indicates whether or not an artificial link has been switched off. For the flow directions and simple states we do not need the ladder variable, since we know that exactly one of them is active at each point in time.

The variables $\delta_{f,t}$ contribute to the objective function for each station with some fixed cost parameter $w_i^f \in [0, 1000]$. The variables $\delta_{s,t}$ contribute to the objective function each with the initialitzation cost $w^s \in \mathbb{N}_{\mathcal{S}}$. The variables $\delta_{a,t}^{\mathrm{on}}$ and $\delta_{a,t}^{\mathrm{off}}$ contribute to the objective function all with the same cost parameter $w^a \in \mathbb{R}_{\geq 0}$, currently $w^a := 5.0$ for all $a \in \mathcal{A}^{\mathrm{ar}}$. Additionally, there is some cost for each compressing link $a \in \mathcal{A}_i^{\mathrm{ar\text{-}co}}$ to be active, i.e., $w_i^{\mathrm{cp\text{-}act}} \in \mathbb{R}_{\geq 0}$.

4.2 Modelling Artificial Links

4.2.1 Bidirected Links

For bidirected links $a \in \mathcal{A}_i^{\mathrm{ar\text{-}bi}}$, we introduce two additional binary variables $x_{a,t}^{\leftarrow}, x_{a,t}^{\rightarrow} \in \{0, 1\}$ for each $t \in \mathcal{T}_0$ which encode the direction of the flow. Thus, we introduce the following constraint

$$x_{a,t}^{\leftarrow} + x_{a,t}^{\rightarrow} = x_{a,t} \quad \forall a \in \mathcal{A}_i^{\mathrm{ar\text{-}bi}}, \forall t \in \mathcal{T}_0 \tag{20}$$

After the direction is chosen, the arcs are modelled like their monodirected counterparts around the corresponding binary variable.

4.2.2 Shortcut

For a shortcut $a = (\ell, r) \in \mathcal{A}_i^{\mathrm{ar\text{-}sc\text{-}bi}}$ the following equations hold:

$$p_{\ell,t} - p_{r,t} \leq (1 - x_{a,t})(\bar{p}_{\ell,t} - \underline{p}_{r,t}) \quad \forall t \in \mathcal{T} \tag{21}$$

$$p_{\ell,t} - p_{r,t} \geq (1 - x_{a,t})(\underline{p}_{\ell,t} - \bar{p}_{r,t}) \quad \forall t \in \mathcal{T} \tag{22}$$

$$q_{a,t}^{\rightarrow} \leq \bar{q}_{a,t} x_{a,t}^{\rightarrow} \qquad \forall t \in \mathcal{T} \tag{23}$$

$$q_{a,t}^{\leftarrow} \leq \bar{q}_{a,t} x_{a,t}^{\leftarrow} \qquad \forall t \in \mathcal{T} \tag{24}$$

If the shortcut is active at some point in time $t \in \mathcal{T}$, i.e., $x_{a,t} = 1$, the pressures at ℓ and r have to be equal and the flow can take an arbitrary direction and value, i.e., there may be forward flow $q_{a,t}^{\rightarrow} \in [0, \bar{q}_{a,t}]$ or backward flow $q_{a,t}^{\leftarrow} \in [0, \bar{q}_{a,t}]$. If the shortcut is not active, the pressure values are decoupled and there is no flow on a.

4.2.3 Regulating Links

For a regulating link $a = (\ell, r) \in \mathcal{A}_i^{\mathrm{ar\text{-}rg}}$, the following equations hold

$$p_{\ell,t} - p_{r,t} \geq (1 - x_{a,t})(\underline{p}_{\ell,t} - \bar{p}_{r,t}) \quad \forall t \in \mathcal{T} \tag{25}$$

$$q_{a,t}^{\rightarrow} \leq \bar{q}_{a,t} x_{a,t} \qquad \forall t \in \mathcal{T} \tag{26}$$

where $q^{\rightarrow}_{a,t} \in [0, \bar{q}_{a,t}]$ is the corresponding forward flow variable. If the regulator is active at some point in time $t \in \mathcal{T}$, i.e., $x_{a,t} = 1$, then the pressure at ℓ has to be greater or equal than the pressure at r. If it is not active, the pressure values are decoupled and there is no flow. For a bidirected regulating link $a = (\ell, r) \in \mathcal{A}^{\text{ar-rg-bi}}_i$, we derive the following corresponding constraints

$$p_{\ell,t} - p_{r,t} \geq (1 - x^{\rightarrow}_{a,t})(\underline{p}_{\ell,t} - \bar{p}_{r,t}) \quad \forall t \in \mathcal{T} \tag{27}$$

$$p_{r,t} - p_{\ell,t} \geq (1 - x^{\leftarrow}_{a,t})(\underline{p}_{r,t} - \bar{p}_{\ell,t}) \quad \forall t \in \mathcal{T} \tag{28}$$

$$q^{\rightarrow}_{a,t} \leq \bar{q}_{a,t} x^{\rightarrow}_{a,t} \quad \forall t \in \mathcal{T} \tag{29}$$

$$q^{\leftarrow}_{a,t} \leq \bar{q}_{a,t} x^{\leftarrow}_{a,t} \quad \forall t \in \mathcal{T} \tag{30}$$

4.2.4   Assigning Machines to Compressing Links

For a link $a = (\ell, r) \in \mathcal{A}^{\text{ar-co}}_i$, which is able to compress gas, there are a number of constraints which depend on the assignment of so-called compressing machines to them.

For each navi station, we are given a set of compressing machines $\mathcal{M}_i := \{m_1, \ldots, m_h\}$ and for each compressing arc $a \in \mathcal{A}^{\text{ar-co}}_i$, there is a subset $\mathcal{M}^a_i \subseteq \mathcal{M}_i$ of machines that can be assigned to it.

Each machine can be assigned cumulatively to at most one compressing link in each timestep $t \in \mathcal{T}$. Hence, we introduce variables $y_{m_j,a,t} \in [0, 1]$ and a constraint

$$\sum_{a \in \mathcal{A}^{\text{ar-co}}_i : m_j \in \mathcal{M}^a_i} y_{m_j,a,t} \leq 1 \quad \forall m_j \in \mathcal{M}_i, \forall t \in \mathcal{T}. \tag{31}$$

Additionally, for each compressing link a maximum number of machines $M^{\max}_a$, that may be assigned to it, is given and we introduce the following constraint

$$\sum_{m_j \in \mathcal{M}^a_i} y_{m_j,a,t} \leq M^{\max}_a x_{a,t} \quad \forall a \in \mathcal{A}^{\text{ar-co}}_i, \forall t \in \mathcal{T}. \tag{32}$$

Further, each compressing machine has an associated maximum compression ratio $R_{m_j,t} > 1$, a maximum power value $P_{m_j,t} \in \mathbb{R}_{\geq 0}$ and a maximum volumetric flow $Q_{m_j,t} \in \mathbb{R}_{\geq 0}$ for each $t \in \mathcal{T}$. Thus, for each compressing link we derive the following three types of constraints

$$\sum_{m_j \in \mathcal{M}^a_i} P_{m_j,t} y_{m_j,a,t} = \tilde{\pi}_{a,t} \quad \forall a \in \mathcal{A}^{\text{ar-co}}_i, \forall t \in \mathcal{T} \tag{33}$$

$$\sum_{m_j \in \mathcal{M}^a_i} Q_{m_j,t} y_{m_j,a,t} = \tilde{q}_{a,t} \quad \forall a \in \mathcal{A}^{\text{ar-co}}_i, \forall t \in \mathcal{T} \tag{34}$$

$$1 + \sum_{m_j \in \mathcal{M}^a_i} (R_{m_j,t} - 1) y_{m_j,a,t} = r_{a,t} \quad \forall a \in \mathcal{A}^{\text{ar-co}}_i, \forall t \in \mathcal{T} \tag{35}$$

The first constraint (33) determines the power that may be used on this link to compress the gas, i.e., $tilde\pi_{a,t}$. The second constraint (56) calculates the maximum amount of flow that can be compressed if the assigned machines run in parallel, i.e., $\tilde{q}_{a,t}$. On the other hand, the third constraint (35) is a (conservative) approximation of

the maximum compression ratio if the assigned machines run in serial. i.e., $r_{a,t}$. It is an approximation, because the compression ratios of serial machines would be multiplied. But since we want to avoid nonlinearities in our formulation, we choose this approximating constraint instead.

Since we assume to have both entities simultaneously at hand, maximum flow and maximum compression ratio at the same time, we probably overestimate the capabilities of the compressing link.

For the three variables introduced above, we add the following additional constraints

$$\tilde{\pi}_{a,t} \geq \pi_{a,t} \qquad \forall a \in \mathcal{A}_i^{\text{ar-co}}, \forall t \in \mathcal{T} \qquad (36)$$

$$\tilde{q}_{a,t} \geq q_{a,t}^{\rightarrow} \qquad \forall a \in \mathcal{A}_i^{\text{ar-co}}, \forall t \in \mathcal{T} \qquad (37)$$

$$p_{\ell,0} r_{a,t} - p_{r,t} \geq (1 - x_{a,t})(p_{\ell,0} - \bar{p}_{r,t}) \quad \forall a \in \mathcal{A}_i^{\text{ar-co}}, \forall t \in \mathcal{T} \qquad (38)$$

The last constraint (38) bounds the outgoing pressure by the maximum compression ratio, while the first two constraints (36) and (37) choose flow or power values within the corresponding bounds, respectively. These values play an important role in the highly nonlinear power bound equation, which we approximate by the following two constraints:

$$\alpha_1 p_{\ell,t} + \alpha_2 p_{r,t} + \alpha_3 q_{a,t}^{\rightarrow} + \alpha_4 \pi_{a,t} \leq \beta + (1 - x_{a,t})(\alpha_1 \underline{p}_{\ell,t} + \alpha_2 \bar{p}_{\ell,t} - \beta) \quad \forall t \in \mathcal{T} \tag{39}$$

$$\alpha_1 p_{\ell,t} + \alpha_2 p_{r,t} + \alpha_3 q_{a,t}^{\rightarrow} + \alpha_4 \pi_{a,t} \geq \beta + (1 - x_{a,t})(\alpha_1 \bar{p}_{\ell,t} + \alpha_2 \underline{p}_{\ell,t} - \beta) \quad \forall t \in \mathcal{T}. \tag{40}$$

Note that $\alpha_2, \alpha_3 \in \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_4 \in \mathbb{R}_{\leq 0}$. To derive the α-coefficients, we sample 10.000 variable triples (p_u, p_v, π_a) and calculate the maximum possible flow $q_a^{\rightarrow}$ using the original power bound equation. With the resulting 4-tuples we run a linear regression algorithm and derive the linear approximation

$$\alpha_1 p_{\ell} + \alpha_2 p_r + \alpha_3 q_a^{\rightarrow} + \alpha_4 \pi_a = \beta.$$

If the corresponding compression arc is active, this equation is enforced by constaints 39 and 40. The power, which is used to compress the gas in a navi station, i.e., $\pi_{a,t}$ contributes with cost $w_i^{\text{cp-val}}$ to the objective function.

4.2.5 Compressing Links

For a directed compressing link $a = (\ell, r) \in \mathcal{A}_i^{\text{ar-cp}}$, the following equations hold:

$$p_{\ell,t} - p_{r,t} \leq (1 - x_{a,t})(\bar{p}_{\ell,t} - \underline{p}_{r,t}) \qquad \forall t \in \mathcal{T} \qquad (41)$$

$$\bar{R}_{a,t} p_{\ell,t} - p_{r,t} \geq (1 - x_{a,t})(\bar{R}_{a,t} \underline{p}_{\ell,t} - \bar{p}_{r,t}) \quad \forall t \in \mathcal{T} \qquad (42)$$

If the compressor is active at some point in time $t \in \mathcal{T}$, i.e., $x_{a,t} = 1$, the pressure at ℓ has to be smaller or equal than the pressure at r. But $p_{r,t}$ can be at most $\bar{R}_{a,t} p_{\ell,t}$ where $\bar{R}_{a,t}$ is the maximum compression ratio of a at time t, which can be determined, for example, from the corresponding constraints (32) and (38). If it is not active, the pressure values are decoupled and there is no flow on a due to constraints (37) and (56).

Further, there may be an additional restriction on the pressure at node r, up to which the pressure can be increased. Let $\bar{p}_{r,t}^{\mathrm{out}}$ be this upper bound. This can be modelled by the following constraint:

$$p_{r,t} \leq \bar{p}_{r,t} - x_{a,t}(\bar{p}_{r,t} - \bar{p}_{r,t}^{\mathrm{out}}) \quad \forall t \in \mathcal{T} \tag{43}$$

For a bidirected compressing arc $a = (\ell, r) \in \mathcal{A}_i^{\mathrm{ar\text{-}cp\text{-}bi}}$, which can compress in forward and backward direction, we derive the following constraints

$$
\begin{aligned}
p_{\ell,t} - p_{r,t} &\leq (1 - x_{a,t}^{\rightarrow})(\bar{p}_{\ell,t} - \underline{p}_{r,t}) && \forall t \in \mathcal{T} && (44) \\
p_{r,t} - p_{\ell,t} &\leq (1 - x_{a,t}^{\leftarrow})(\bar{p}_{r,t} - \underline{p}_{\ell,t}) && \forall t \in \mathcal{T} && (45) \\
\bar{R}_{a,t}p_{\ell,t} - p_{r,t} &\geq (1 - x_{a,t}^{\rightarrow})(\bar{R}_{a,t}\underline{p}_{\ell,t} - \bar{p}_{r,t}) && \forall t \in \mathcal{T} && (46) \\
\bar{R}_{a,t}p_{r,t} - p_{\ell,t} &\geq (1 - x_{a,t}^{\leftarrow})(\bar{R}_{a,t}\underline{p}_{r,t} - \bar{p}_{\ell,t}) && \forall t \in \mathcal{T} && (47) \\
q_{a,t}^{\rightarrow} &\leq x_{a,t}^{\rightarrow}\bar{q}_{a,t} && \forall t \in \mathcal{T} && (48) \\
q_{a,t}^{\leftarrow} &\leq x_{a,t}^{\leftarrow}\bar{q}_{a,t} && \forall t \in \mathcal{T} && (49) \\
p_{r,t} &\leq \bar{p}_{r,t} - x_{a,t}^{\rightarrow}(\bar{p}_{r,t} - \bar{p}_{r,t}^{\mathrm{out}}) && \forall t \in \mathcal{T} && (50) \\
p_{\ell,t} &\leq \bar{p}_{\ell,t} - x_{a,t}^{\leftarrow}(\bar{p}_{\ell,t} - \bar{p}_{\ell,t}^{\mathrm{out}}) && \forall t \in \mathcal{T} && (51)
\end{aligned}
$$

Furthermore, the constraints explained in Section 4.2.4 regarding the power, compression ratio, volumetric flow, and machine assignment obviously apply to compressing links.

4.2.6 Combined Links

A combined link $a = (\ell, r) \in \mathcal{A}_i^{\mathrm{ar\text{-}co}}$ can work as regulating link or a compressing link. Hence, first of all we introduce two decision variables, which encode in which mode the combined link is running.

$$x_{a,t}^{\mathrm{rg}} + x_{a,t}^{\mathrm{cp}} = x_{a,t} \quad \forall a \in \mathcal{A}_i^{\mathrm{ar\text{-}bi}}, \forall t \in \mathcal{T}_0 \tag{52}$$

For the regulating mode, we get one additional constraint regarding the pressur coupling

$$p_{\ell,t} - p_{r,t} \geq (1 - x_{a,t}^{\mathrm{rg}})(\underline{p}_{\ell,t} - \bar{p}_{r,t}) \quad \forall t \in \mathcal{T}, \tag{53}$$

while for the compressing mode we have

$$
\begin{aligned}
p_{\ell,t} - p_{r,t} &\leq (1 - x_{a,t}^{\mathrm{cp}})(\bar{p}_{\ell,t} - \underline{p}_{r,t}) && \forall t \in \mathcal{T} && (54) \\
\bar{R}_{a,t}p_{\ell,t} - p_{r,t} &\geq (1 - x_{a,t}^{\mathrm{cp}})(\bar{R}_{a,t}\underline{p}_{\ell,t} - \bar{p}_{r,t}) && \forall t \in \mathcal{T}. && (55)
\end{aligned}
$$

Additionally, the constraints explained in Section 4.2.4 regarding the power, compression ratio, volumetric flow, and machine assignment apply as well (replacing variable $x_{a,t}$ by $x_{a,t}^{\mathrm{cp}}$ of course), except for (37), which has to be changed to

$$\bar{q}_{a,t}x_{a,t}^{\mathrm{rg}} + \tilde{q}_{a,t} \geq q_{a,t}^{\rightarrow} \quad \forall t \in \mathcal{T} \tag{56}$$

in order to allow for flow in the regulating mode.

4.3 Flow Directions Constraints

4.3.1 In- and Outflow Constraints

For each fence node $v \in \mathcal{V}_i$ and each point in time $t \in \mathcal{T}$ we introduce a variable $q_{v,t}^{\mathrm{fn}}$ counting the total in- or outflow from the remaining network

$$\sum_{(\ell,v)\in\mathcal{A}^{\mathrm{ar}}} q_{a,t}^{\rightarrow} - \sum_{(v,r)\in\mathcal{A}^{\mathrm{ar}}} q_{a,t}^{\rightarrow} + \sum_{(v,r)\in\mathcal{A}^{\mathrm{ar\text{-}bi}}} q_{a,t}^{\leftarrow} - \sum_{(\ell,v)\in\mathcal{A}^{\mathrm{ar\text{-}bi}}} q_{a,t}^{\leftarrow} = -q_{v,t}^{\mathrm{fn}} \qquad (57)$$

Next, for a flow direction $f = (f^+, f^-) \in \mathcal{F}_i$ we introduce the following two types of constraints:

$$-q_{v,t}^{\mathrm{fn}} + M_v x_{f,t} \leq M_v + \varepsilon \quad \forall f \in \mathcal{F}_i, \forall v \in V_i \setminus f^-, \forall t \in \mathcal{T} \qquad (58)$$

$$q_{v,t}^{\mathrm{fn}} + M_v x_{f,t} \leq M_v + \varepsilon \quad \forall f \in \mathcal{F}_i, \forall v \in V_i \setminus f^+, \forall t \in \mathcal{T}. \qquad (59)$$

Here, M_v is an upper bound on the total outflow and inflow (for example the sum of the upper flow bounds of all incident pipes) and ε currently is $5.000 \frac{m^3}{h}$. By the construction of the constraints, for each entry fence node $v \in f^+$ we have $q_{v,t}^{\mathrm{fn}} \in [-\varepsilon, M_v]$ and for each exit fence node $v \in f^-$ we have $q_{v,t}^{\mathrm{fn}} \in [-M_v, \varepsilon]$, while for all other fence nodes we have $q_{v,t}^{\mathrm{fn}} \in [-\varepsilon, \varepsilon]$.

4.3.2 Flow Direction Conditions

For each navi station, there exist so-called flow direction conditions $\mathcal{W}_i$. These conditions on the in- and outflows at fence nodes have to be satisfied, if a certain flow direction shall be active.

To define them formally, let $f = (f^+, f^-) \in \mathcal{F}_i$ be a flow direction and let $\mathcal{Q} := \{\geq, \leq\}$ be the set of supported mathematical operators. Currently, there are two types of flow directions conditions that we support:

- Flow conditions $c := (f, \mathcal{V}_i^{c_1}, \sim, M_c) \in \mathcal{F}_i \times \mathcal{P}(\mathcal{V}_i) \times \mathcal{Q} \times \mathbb{R}_{\geq 0}$ and

- comparison conditions $c := (f, \mathcal{V}_i^{c_1}, \sim, \mathcal{V}_i^{c_2}) \in \mathcal{F}_i \times \mathcal{P}(\mathcal{V}_i) \times \mathcal{Q} \times \mathcal{P}(\mathcal{V}_i)$.

Thus, $\mathcal{W}_i \subseteq (\mathcal{F}_i \times \mathcal{P}(\mathcal{V}_i) \times \mathcal{Q} \times \mathbb{R}_{\geq 0}) \cup (\mathcal{F}_i \times \mathcal{P}(\mathcal{V}_i) \times \mathcal{Q} \times \mathcal{P}(\mathcal{V}_i))$.

For each $c \in \mathcal{W}_i$ we have that either $\mathcal{V}_i^{c_1} \subseteq f^+$ or $\mathcal{V}_i^{c_1} \subseteq f^-$ and analogously that either $\mathcal{V}_i^{c_2} \subseteq f^+$ or $\mathcal{V}_i^{c_2} \subseteq f^-$ in case of a comparison condition. Next, given a flow direction and a fence node, the function sgn returns 1, if the node is an entry w.r.t. the flow direction, -1, if it is an exit, and 0 otherwise:

$$\mathrm{sgn} : \mathcal{F}_i \times \mathcal{V}_i \to \{-1, 0, 1\}, (f, v) \to \begin{cases} 1 & \text{if } v \in f^+ \\ -1 & \text{if } v \in f^- \\ 0 & \text{otherwise.} \end{cases}$$

Using the definitions from above, for each $t \in \mathcal{T}$ we can write the flow conditions as follows

$$\sum_{v\in\mathcal{V}_i^{c_1}} \mathrm{sgn}(f,v) q_{v,t}^{\mathrm{fn}} \geq -\sum_{v\in\mathcal{V}_i^{c_1}} M_v + x_{f,t}(M_c + \sum_{v\in\mathcal{V}_i^{c_1}} M_v) \quad \forall(f, \mathcal{V}_i^{c_1}, \geq, M_c) \in C^i$$

$$(60)$$

$$\sum_{v\in\mathcal{V}_i^{c_1}} \mathrm{sgn}(f,v) q_{v,t}^{\mathrm{fn}} \leq \sum_{v\in\mathcal{V}_i^{c_1}} M_v + x_{f,t}(M_c - \sum_{v\in\mathcal{V}_i^{c_1}} M_v) \quad \forall(f, \mathcal{V}_i^{c_1}, \leq, M_c) \in C^i$$

$$(61)$$

while the comparison conditions can be written as

$$\sum_{v \in \mathcal{V}_i^{c1}} \text{sgn}(f,v) q_{v,t}^{\text{fn}} - \sum_{v \in \mathcal{V}_i^{c2}} \text{sgn}(f,v) q_{v,t}^{\text{fn}} \geq$$

$$- \sum_{v \in \mathcal{V}_i^{c1} \cup \mathcal{V}_i^{c2}} M_v + x_{f,t} \sum_{v \in \mathcal{V}_i^{c1} \cup \mathcal{V}_i^{c2}} M_v \qquad \forall (f, \mathcal{V}_i^{c1}, \geq, \mathcal{V}_i^{c2}) \in C^i \qquad (62)$$

$$\sum_{v \in \mathcal{V}_i^{c2}} \text{sgn}(f,v) q_{v,t}^{\text{fn}} - \sum_{v \in \mathcal{V}_i^{c1}} \text{sgn}(f,v) q_{v,t}^{\text{fn}} \geq$$

$$- \sum_{v \in \mathcal{V}_i^{c1} \cup \mathcal{V}_i^{c2}} M_v + x_{f,t} \sum_{v \in \mathcal{V}_i^{c1} \cup \mathcal{V}_i^{c2}} M_v \qquad \forall (f, \mathcal{V}_i^{c1}, \leq, \mathcal{V}_i^{c2}) \in C^i \qquad (63)$$

4.3.3 Exit Pressure Constraints

For some fence nodes $v \in \mathcal{V}_i$, there is an additional upper (tighter) pressure bound $\bar{p}_v^{\text{exit}}$ if a flow direction $f = (f^+, f^-) \in \mathcal{F}_i$ is chosen, where v is an exit fence node, i.e., $v \in f^-$. We can model this condition via the following constraint:

$$p_{v,t} \leq \bar{p}_{v,t} + x_{f,t}(\bar{p}_v^{\text{exit}} - \bar{p}_{v,t}) \quad \forall f \in \mathcal{F}_i \text{ with } v \in f^-, \forall t \in \mathcal{T}. \qquad (64)$$

5 Connecting Variables and Constraints

5.1 Massflow Conservation Equation

For all nodes $v \in \mathcal{V}$ we introduce the so-called massflow conservation equations. For each inner node $v \in \mathcal{V}^0$ and each timestep $t \in \mathcal{T}$ the amount of flow entering v has to leave it, too. Thus, for each inner node $v \in \mathcal{V}^0$ we have

$$\sum_{a=(\ell,v) \in \mathcal{A}^{\text{pi}}} q_{v,t,a} - \sum_{a=(v,r) \in \mathcal{A}^{\text{pi}}} q_{v,t,a}$$

$$+ \sum_{a=(\ell,v) \in \mathcal{A}^{\text{rg}}} q_{a,t} - \sum_{a=(v,r) \in \mathcal{A}^{\text{rg}}} q_{a,t}$$

$$+ \sum_{x=(\ell,v) \in \mathcal{A}^{\text{va}}} q_{x,t} - \sum_{x=(v,r) \in \mathcal{A}^{\text{va}}} q_{x,t}$$

$$+ \sum_{(\ell,v) \in \mathcal{A}^{\text{ar}}} \overrightarrow{q_{a,t}} - \sum_{(v,r) \in \mathcal{A}^{\text{ar}}} \overrightarrow{q_{a,t}}$$

$$+ \sum_{(v,r) \in \mathcal{A}^{\text{ar-bi}}} \overleftarrow{q_{a,t}} - \sum_{(\ell,v) \in \mathcal{A}^{\text{ar-bi}}} \overleftarrow{q_{a,t}} = 0 \quad \forall v \in \mathcal{V}^0, \ \forall t \in \mathcal{T}. \qquad (65)$$

For a boundary node $v \in \mathcal{V}^{\text{b}}$, the supply or demand must be satisfied. Hence, for each boundary node $v \in \mathcal{V}^{\text{b}}$ we have

$$\sum_{a=(\ell,v) \in \mathcal{A}^{\text{pi}}} q_{v,t,a} - \sum_{a=(v,r) \in \mathcal{A}^{\text{pi}}} q_{v,t,a}$$

$$+ \sum_{a=(\ell,v) \in \mathcal{A}^{\text{rg}}} q_{a,t} - \sum_{a=(v,r) \in \mathcal{A}^{\text{rg}}} q_{a,t}$$

$$+ \sum_{x=(\ell,v) \in \mathcal{A}^{\text{va}}} q_{x,t} - \sum_{x=(v,r) \in \mathcal{A}^{\text{va}}} q_{x,t} = d_{v,t} \quad \forall v \in \mathcal{V}^{\text{b}}, \ \forall t \in \mathcal{T}. \qquad (66)$$

Since boundary nodes are not part of navi stations by definition, i.e., $\mathcal{V}_i \subseteq \mathcal{V}^0$ for all $i \in \{1..m\}$, we do not need to consider flows of artificial links for the ladder constraints.

6   The Netmodel-Algorithm

Currently, the algorithm for solving the Netmodel MIP works as follows:

6.1   Stage NO_SLACKS

In the first stage we try to solve the Netmodel MIP without using any slack. This is done by fixing the slack variables on the demand and the inflow pressures to zero, in particular:

$$\sigma_{v,t}^{d+} = \sigma_{v,t}^{d-} = \sigma_{v,t}^{p+} = \sigma_{v,t}^{p-} = 0 \quad \forall v \in \mathcal{V}^{\mathrm{b}} \text{ and } \forall t \in T. \tag{67}$$

We run the MIP with a time limit of 3600 seconds. If any feasible solution is found within this time limit, we tag the instance with NO_SLACKS and go into the Iterative Velocity Adjustment Procedure in Section 6.4. Otherwise, we proceed with Stage FLOW_SLACKS.

6.2   Stage FLOW_SLACKS

In the second stage we try to solve the Netmodel MIP without using any inflow pressure slacks. This is done by fixing the slack variables on the inflow pressures to zero, in particular:

$$\sigma_{v,t}^{p+} = \sigma_{v,t}^{p-} = 0 \quad \forall v \in \mathcal{V}^{\mathrm{b}} \text{ and } \forall t \in T. \tag{68}$$

Note that demand/supply slacks are allowed now, i.e, the corresponding variables are not fixed to zero. We run the MIP with a time limit of 3600 seconds. If any feasible solution is found within this time limit, we tag the instance with FLOW_SLACKS and go to the Iterative Velocity Adjustment Procedure in Section 6.4. Otherwise, we proceed with Stage FLOW_AND_PRESSURE_SLACKS.

6.3   Stage FLOW_AND_PRESSURE_SLACKS

In the third stage we try to solve the Netmodel MIP as described in this document without any modifications or fixations. Note that demand/supply- and inflow pressure slacks are both allowed now. We run the MIP with a time limit of 3600 seconds. If any feasible solution is found within this time limit, we tag the instance with FLOW_AND_PRESSURE_SLACKS go to the Iterative Velocity Adjustment Procedure in Section 6.4. Otherwise, we terminate the algorithm and tag the instance as INFEASIBLE.

6.4   Iterative Velocity Adjustment Procedure

As mentioned in Section 3.1, in the initial Netmodel MIP we fix the absolute velocity in the friction term of the momentum equation (6) to the absolute velocity at the corresponding node v at time 0 for all timesteps. This is because up to

this point we have no better estimate of what the velocity is going to be like in the future. In order to improve the quality of our solution, given the best solution sol_0 from the previous stages, using the flow and pressure values at v at a certain timestep $t \in \mathcal{T}$, i.e., $q_{a,t,v}$ and $p_{v,t}$, we can calculate an updated (better?!) absolute velocity $|v_{v,t}|$ value for the future.

The idea now is to create a new MIP*, where we fix we fix the binary variables of the flow directions to the values in sol_0, update the (6) with the newly calculated absolute velocities, and resolve MIP* in the stage of the initial MIP. Thereby we hope, that the assumed velocities and the velocities we can retrieve from a solution lie closer together and the solution quality improves.

In order to avoid "cycling" between solutions, we repeat this procedure k times and use the average absolute velocity of the last j runs instead of only the last one. The final solution after this iteration process is then passed on to the smoothing stage. The complete Iterative Velocity Adjustment Procedure, with all its different cases, that may happen, is described in Algorithm 1

Algorithm 1 Iterative Velocity Adjustment Procedure

1: $\text{sol}_0 \leftarrow$ solution of initial MIP
2: MIP$^* \leftarrow$ initial MIP with flow directions fixed as in sol_0
3: tag$^* \leftarrow$ tag of initial MIP
4:
5: **for** i in $1 \ldots k$ **do**
6:     determine average gas velocities of last $\min\{i, j\}$ solutions
7:     update constraints (6)
8:     resolve MIP*
9:     **if** MIP* is infeasible **then**
10:         **if** tag* = NO_SLACKS **then**
11:             tag$^* \leftarrow$ FLOW_SLACKS
12:             Add slack on supply/demands and go to 9
13:         **end if**
14:         **if** tag* = FLOW_SLACKS **then**
15:             tag$^* \leftarrow$ FLOW_AND_PRESSURE_SLACKS
16:             Add slack on pressure bounds and go to 9
17:         **end if**
18:         **if** tag* = FLOW_AND_PRESSURE_SLACKS **then**
19:             **return** sol_{i-1}
20:         **end if**
21:     **else**
22:         $\text{sol}_i \leftarrow$ solution of MIP*
23:     **end if**
24: **end for**
25:
26: **return** sol_k

6.5 Smoothing Stage

Due to the nature of many MIP solutions, it may happen, that in a solution the pressure and flow values in the navi stations differ a lot from one time step to

the next. In order to avoid this behaviour, we introduce the following smoothing procedure as a postprocessing step.

Given a solution from the Iterative Velocity Adjustment Procedure, we fix all binary variables to their solution values. Afterwards, we add some smoothing constraints and variables to the model and solve the resulting LP. The idea is to smooth the in- and outflow values and the pressures at the fence group nodes over time, i.e., we try to change these values as few as possible while still operating in the suggested flow directions, simple states, etc. Hence, for each navi station G_i, each node $v \in \mathcal{V}_i^{\mathrm{fn}}$, and each $t \in \mathcal{T}$ we add four variables $\delta_{v,t}^{q+}, \delta_{v,t}^{q-} \in \mathbb{R}^+$ and $\delta_{v,t}^{p+}, \delta_{v,t}^{p-} \in \mathbb{R}^+$, while for all artificial nodes $v \in \mathcal{V}_i^{\mathrm{ar}}$ we only add the ladder ones. The smoothing constraints on the nodes are then the following.

$$p_{v,t} - p_{v,t-1} + \delta_{v,t}^{p+} - \delta_{v,t}^{p-} = 0 \quad \forall v \in \mathcal{V}_i, \forall t \in \mathcal{T} \tag{69}$$

$$q_{v,t}^{\mathrm{fn}} - q_{v,t-1}^{\mathrm{fn}} + \delta_{v,t}^{q+} - \delta_{v,t}^{q-} = 0 \quad \forall v \in \mathcal{V}_i^{\mathrm{fn}}, \forall t \in \mathcal{T} \tag{70}$$

Both types, the pressure smoothing and the flow smoothing variables, contribute to the objective function with some value $w^{\mathrm{sm\text{-}p}}$ and $w^{\mathrm{sm\text{-}q}}$, respectively.

If the smoothing LP is infeasible (may happen due to numerical issues when fixing the binary variables), we return the input solution is used (probably not very smooth) and add an _SI to the tag of the last MIP*. Otherwise, simply the tag of the last MIP* is returned.