Competitive Imbalance of Heterogeneous Teams in Closed Leagues and Dominance Criteria

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Abstract

In this article we explore a new application about sequential dominance (introduced in the fields of social welfare measurement): The measurement of competitive imbalance in sports leagues when observations of both teams' points and fan-bases are available. We argue that desirable properties for the measurement of imbalance – *e.g.* a dual concept of the inclination to any increase in downside risk – lead us to present a new sequential dominance criterion.

JEL Classification: D63 ; L83 ; Z28. *Keywords*: Sequential dominance ; Competitive imbalance.

1 Introduction

Since the seminal articles of Rottenberg (1956) and Neale (1964), a significant part of the literature devoted to professional sports leagues refers to the question of competitive imbalance. They point out that, in an unbalanced league, spectators' interest is low because the results are predictable. Since a high level of imbalance is supposed to depress demand and confidence in the legitimacy of the competition, the major part of the literature has been concerned with mechanisms (revenue sharing agreements, salary caps, luxury taxes, draft, etc.) which may restore a significant level of competitive imbalance (El-Hodiri and Quirk 1971; Quirk and El-Hodiri 1974; Fort and Quirk 1995; Vrooman 1995; Kesenne 2000a, 2000b; Szymanski and Kesenne 2004; Feess and Stahler 2009 etc.). A minor part of the literature is dedicated to the question of the appropriate measure of competitive imbalance. Since

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there is an obvious analogy between measuring competitive imbalance and income inequality, Schmidt and Berri (2001, 2002) and Utt and Fort (2002) resort to the Gini coefficient and Horowitz (1997), Borooah and Mangan (2012) and Gayant and Le Pape (2017) opt for the use of entropy indices. Gayant and Le Pape (2017) point out that, in spite of the analogy between measuring competitive imbalance and income inequality, the main concern when studying competitive imbalance is the 'attractiveness' of a league, which does not coincide with the issue of 'fairness' in a society. The central claim of their approach is that, whereas it seems relevant to request that the Pigou-Dalton principle of transfers should be fulfilled in the field of imbalance measurement, the axiom needed at the next order is the exact opposit of the Shorrocks and Foster transfer sensitivity principle (Kolm 1976; Shorrocks and Foster 1987 ; Davies and Hoy 1994, 1995 ; Chiu 2007). Indeed, a league is more attractive if it is more balanced among leading teams, that is, at the top of the distribution of points.

Unlike what exists in the field of income inequality measurement, to the best of our knowledge, there is no work providing dominance criteria to the measurement of competitive imbalance. The dominance criteria are consistent with classes of measures so that they limit the arbitrariness of the choice of one measure rather than another. The Lorenz dominance criterion of Atkinson (1970) is well known, and is widely used for making comparisons on the basis of income distribution data. However, this approach, as well as those in the field of competitive imbalance measurement, do not take into account non-income information and non-point information, respectively. Non-income information – such as family composition – are available in micro data-sets and may be relevant for the measurement income inequality. Hence, from the results of Atkinson (1970), it is not possible to state a transfer from single persons to families with children as inequality reducing. Likewise, non-point information – such as teams' fan-bases – are available and may be relevant for the measurement of competitive imbalance. The idea is that, at any point level, fans' satisfaction a team with a larger fan-base manages to achieve is at most as high as fans' satisfaction attained by a team with a smaller fan-base. The results of Borooah and Mangan (2012) and Gayant and Le Pape (2017), among others, do not allow for recommending as attractive the transfer of points from teams with small fan-bases to teams with large fan-bases. In response to this limitation, Atkinson and Bourguignon (1987) introduce the sequential Lorenz criterion, for the comparison of joint distributions of income and needs. Subsequently, Jenkins and Lambert (1993) and Moyes (2012) among others, generalize this approach to allow for demographic change. Lambert and Ramos (2002) extend the approach of Atkinson and Bourguignon (1987) and propose a criterion consistent with the Shorrocks and Foster transfer sensitivity principle.

In this article, we adapt the approach of Atkinson and Bourguignon (1987) to the measurement of competitive imbalance. Since professional leagues are composed by a finite number of teams that can obtain a finite number of points, the sequential dominance criteria we present are defined on grids. According to the criteria, a league is more attractive if it is more balanced among teams with larger fan-bases. We extend the approach of Atkinson and Bourguignon (1987), which is consistent with the principle of transfers of Pigou-Dalton, in the exact opposite sense than that of Lambert and Ramos (2002). We introduce a downward dominance criterion, which is in line with the central claim of Gayant and Le Pape (2017). The criterion deals with dominance curves that are aggregated from above, *e.g.* from the highest point level, as Aaberge (2009) does in an unidimensional framework. According to the criterion, a league is more attractive if it is more balanced among leading teams with larger fan-bases, that is, at the top of the conditional distribution of points among teams with larger fan-bases.

Section 2 introduces the framework and the basic result of Atkinson and Bourguignon (1987) when variables are defined on grids. Section 3 exhibits another result of Atkinson and Bourguignon (1987) and explains its suitability in the field of competitive imbalance. Section 4 presents our main result. Section 5 concludes the article.

2 Formalization and hypothetical configurations

We investigate a heterogeneous population of N teams. There are T types i of teams, $2 \leqslant T \leqslant N$, having different fan-bases so that the team types can be ranked by decreasing popularity: teams of type 1 have the greater fan-base, teams of type 2 have the second greater fan-base and so on up to teams of type T that have the smaller fan-base. We face T homogeneous sub-populations, each one consists of n_i teams, $i = 1, \ldots, T$ so that the total number of teams may be expressed by $N = \sum_{i=1}^{T} n_i$.

The contest is supposed to be a "one home-one away" closed championship (*i.e.* without promotion and relegation), in which each team plays $2(N-1)$ games and the total number of games played is $N(N-1)$. As Gayant and Le Pape (2017), we choose to measure the degree of imbalance of the league on the distribution of points rather than on the distribution of wins to make our results also suitable for sports in which ties can happen. Suppose that, for every game that it plays, a team is awarded z_w points for a win, z_t for a tie, and z_ℓ points for a loss ($z_w > z_t > z_\ell$). For the sake of simplicity, we choose a particular point award system which fulfills the condition $2z_t = z_w + z_\ell$: From here on, we assume that $z_w = 2$; $z_t = 1$; $z_\ell = 0$.

The points obtained at the end of the contest by any team lie in a finite set of integers:

$$\mathscr{P} = \{p_0 + j \ : \ j = 0, 1, \ldots, 4N-4\}.$$

According to the "Perfect Competitive Imbalance" hypothetical configuration defined by

Gayant and Le Pape (2017), a team cannot obtain less than $p_0 = 0$ point and it cannot obtain more than $4N - 4$ points at the end of the contest.[1] The set $\mathscr{P}$ is a subset of $\mathscr{G}$ introduced in Fishburn and Lavalle (1995) in which the separation parameter is equal to 1. For convenience, $p_0 + j$ will sometimes be denoted by p_j, then $p_{j+1} > p_j$ for $j = 0, \ldots, 4N - 5$.

2.1 Social valuation of points and global attractiveness function

The social valuation of points p_j, received by a team of type i, is denoted by $v_i(p_j)$. For future reference we define two classes of functions as follows:

$$\begin{aligned} V_1 =& \{v_i : \mathscr{P} \to \mathbb{R} \ : \ \Delta_1 v_i(p_j) > 0 \ \forall j, j = 0, \ldots, 4N-5, \ \forall i, i = 1, \ldots, T\}; \\ V_2 =& \{v_i \in V_1 \ : \ \Delta_2 v_i(p_j) < 0 \ \forall j, j = 0, \ldots, 4N-6, \ \forall i, i = 1, \ldots, T\} \end{aligned}$$

in which

$$\Delta_1 v_i(p_j) = v_i(p_{j+1}) - v_i(p_j) \text{ and } \Delta_2 v_i(p_j) = \Delta_1 v_i(p_{j+1}) - \Delta_1 v_i(p_j)$$

so functions v_i in V_2 increase at a decreasing rate over $\mathscr{P}$.

The notion of type only makes sense when the objective of the team is one of satisfying its fan-base. The idea is that one team type is lower than another if, at any point level, the satisfaction it manages to achieve is at most as high as the satisfaction attained by the other type. Formally,

$$v_i(p_j) \leqslant v_{i+1}(p_j) \ \ \forall i \in \{1, \ldots, T-1\} \text{ and } \forall j \in \{0, \ldots, 4N-4\}. \tag{1}$$

Hence, at any point level, we consider that the satisfaction of a greater fan-base is lower than, or at least as great as the satisfaction of a smaller fan-base. The type is then an ordinal variable if we consider that only comparisons of levels of satisfaction make sense.

We suppose that the marginal distribution of types is fixed: points are the only attribute whose distribution can be altered in order to decrease or increase attractiveness.[2] In other

[1] The idea of the Perfect Competitive Imbalance is very intuitive: The weakest team loses against all others, the second weakest wins against the weakest and loses against all others, and so on up to the strongest team that wins every games. Under a regular schedule, it can then be objectified as: The first team loses its $2(N-1)$ games, the second team wins 2 games and loses $2(N-2)$ games, ..., the N^{th} team wins its $2(N-1)$ games. Hence, the first team does not obtain any point while the last team obtains $4N-4$ points at the end of the contest.

[2] This constraints rules out the possibility of comparing two leagues with different distributions of types, *e.g.* two leagues from different countries *etc.* However, by using joint distribution functions in lieu of conditional distribution function as we do here, it seems possible to generalize our framework so that leagues with different distributions of types are comparable.

words, we are interested in comparing changes in the conditional distribution of p_j, given types, denoted by $f^i(p_j)$ defined on the following set:

$$\Omega_i = \left\{ f^i(p_j) : \mathscr{P} \to \{0, \frac{1}{n_i}, \ldots, 1\} \ : \sum_{j=0}^{4N-4} f^i(p_j) = 1 \text{ for } j = 0, \ldots, 4N-4 \text{ and } i = 1, \ldots, T \right\}.$$

We use f to denote the distributions $f^i(p_j)$ for $i = 1, \ldots, T$, hence f is defined on the following set:

$$\Omega = \left\{ \begin{array}{l} f(p_j, i) : \mathscr{P} \times \{1, \ldots, T\} \to \{0, \frac{1}{N}, \ldots, 1\} \ : \sum_{i=1}^{T} \sum_{j=0}^{4N-4} f(p_j, i) = 1, \\ \text{for } j = 0, \ldots, 4N-4 \text{ and } i = 1, \ldots, T \end{array} \right\}.$$

As implicitly supposed by Gayant and Le Pape (2017), here the global attractiveness function is additively separable:[3]

$$A_f = \sum_{i=1}^{T} n_i \sum_{j=0}^{4N-4} v_i(p_j) f^i(p_j).$$

If we consider two alternative distributions, f and g in Ω, and denote the difference by $\Delta f = f - g$, then the difference in attractiveness is

$$\Delta A_f = \sum_{i=1}^{T} n_i \sum_{j=0}^{4N-4} v_i(p_j) \Delta f^i(p_j).$$

This may be written as:

$$\Delta A_f = \sum_{i=1}^{T} n_i \Delta A_f^i$$

in which ΔA_f^i denotes the change in attractiveness for a given type i. If we consider the

[3]Gayant and Le Pape (2017) argue for a family of Generalized Entropy indices to measure the attractiveness of a league. This family of indices relies on additively separable functions as introduced in this framework. There are endless debates about the question of whether income inequality should be measured by rank-dependent measures or by measures that take into account the difference in incomes. In our view, the normative measurement of competitive imbalance should be measured by the latter measures. We assert that inequality-reducing transfers are more valuable when taking place between two teams with more points than between two teams with less points, provided the difference in points between the two latter and the two former is the same. On the contrary, we could not assert with certainty that the same applies when their differences in points is different, even if the difference in ranks between the two former teams and the two latter teams is the same.

attractiveness of a particular type of teams i, applying Abel's lemma, we have then:[4]

$$\Delta A^i_f = \sum_{j=0}^{4N-5} \Delta F^i(p_j)[v_i(p_j) - v_i(p_{j+1})] + \Delta F^i(p_{4N-4})v_i(p_{4N-4})$$

in which F^i denotes the cumulative conditional distribution (and $\Delta F^i \equiv F^i - G^i$). Making use of the fact that $\Delta F^i(p_{4N-4}) = 0$, we obtain:

$$\begin{aligned}\Delta A^i_f &= \sum_{j=0}^{4N-5} \Delta F^i(p_j)[v_i(p_j) - v_i(p_{j+1})] \\ &= \sum_{j=0}^{4N-5} \Delta D^1_i(p_j)\Delta_1 v_i(p_j)\end{aligned} \tag{2}$$

in which $\Delta D^1_i(p_j) = -\Delta F^i(p_j)$.

2.2 The interrelation between points and types

The favourable permutation of order 1 we introduce below is the foundation of the approach of Atkinson and Bourguignon (1987), it is inspired from the illustration of Moyes (2012, p. 1360-1).

Definition 2.1. *Given two configurations f and g belonging to Ω. We say that f is obtained from g by means of a favourable permutation of order 1 if there exist two types $i \in \{1, \dots, T-1\}$ and $\ell \in \{2, \dots, T\}$ so that $i < \ell$, and two point levels $p_k \in \{0, \dots, 4N-5\}$ and $p_{k+\delta} \in \{1, \dots, 4N-4\}$ so that $p_k < p_{k+\delta}$ (equivalently, $\delta > 0$) such that:*

$$f^i(p_k) = g^i(p_k) - \frac{1}{n_i} \ ; \ f^i(p_{k+\delta}) = g^i(p_{k+\delta}) + \frac{1}{n_i} \ ;$$
$$f^\ell(p_k) = g^\ell(p_k) + \frac{1}{n_\ell} \ ; \ f^\ell(p_{k+\delta}) = g^\ell(p_{k+\delta}) - \frac{1}{n_\ell} \ ;$$

f and g are identical everywhere else. Equivalently we say that g is obtained from f by means of an unfavourable permutation of order 1.

As an illustration, a favourable permutation of order 1 may be:

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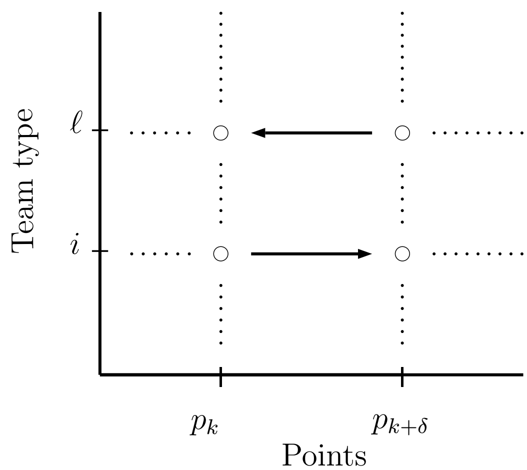
Lemma 2.1. *Abel's summation formula. Let $b_1, \dots, b_N, c_1, \dots, c_N$ be real numbers. Set $B_j = \sum_{k=1}^{j} b_k$. Then:*

$$\sum_{j=1}^{N} b_j c_j = \sum_{j=1}^{N-1} B_j(c_j - c_{j+1}) + B_N c_N.$$



The key feature in this subsection is the variation of social marginal valuation of points, $v_i(p_{j+1})-v_i(p_j)$, across different types. The critical role may be seen simply by considering the case in which $v_i(p_{j+1})-v_i(p_j) = v(p_{j+1})-v(p_j)$ for all $i = 1, \ldots, T$ and for all $j = 0, \ldots, 4N-5$. In that situation, we have only to consider the marginal distribution of points as in Gayant and Le Pape (2017).

We suppose now that there is a limited degree of agreement about the relation between points and types, in the sense that we can rank the types $i = 1$ to T in order such that for all $j = 0, \ldots, 4N-4$, the social marginal valuation of points is non-increasing with i. In other words, at any given point level, teams of type 1 have the highest social marginal valuation of points. We may write this assumption as follows:

$$\text{(H1)} \qquad v_i(p_{j+1}) - v_i(p_j) = \sum_{k=i}^{T} \epsilon_k(p_j)$$

in which $\epsilon_k(p_j) \geqslant 0$ for all $j = 0, \ldots, 4N-5$ and $i = 1, \ldots, T$. So $\epsilon_T(p_j)$ is the social marginal valuation of points for teams of type T with p_j; $\epsilon_T(p_j) + \epsilon_{T-1}(p_j)$ is that for teams of type $T-1$ and so on.

Making use of (2), we have:

$$\Delta A_f = \sum_{j=0}^{4N-5} \sum_{i=1}^{T} n_i [v_i(p_{j+1}) - v_i(p_j)] \Delta D_i^1(p_j). \tag{3}$$

The restriction $v_i \in V_1$ implies that the attractiveness of a league does not decrease as the result of an increase in points from any team. However, the point award system we choose in our framework makes all possible configurations having the same points total, which is $2N(N-1)$. Hence, all admissible point changes combine increases and decreases in points so that the points total remains unchanged. It is then relevant to assert that the restriction

$v_i \in V_1$ together with (H1) implies that the attractiveness of a league does not decrease as the result of a favourable permutation of order 1, as stated in the following proposition.[5]

Proposition 2.1. *The attractiveness of a league does not decrease as the result of a favourable permutation of order* 1 *if, and only if,* $v_i \in V_1$ *and (H1) hold.*

Proof. See in Appendix. □

The assumption (H1) implies that:

$$\Delta A_f = \sum_{j=0}^{4N-5} \left[\epsilon_T(p_j) \sum_{i=1}^{T} n_i \Delta D_i^1(p_j) + \epsilon_{T-1}(p_j) \sum_{i=1}^{T-1} n_i \Delta D_i^1(p_j) + \ldots + \epsilon_1(p_j) n_1 \Delta D_1^1(p_j) \right].$$

From this, we can see that a sufficient condition for $\Delta A_f \geqslant 0$ is that:

$$(C_1^*) \qquad \sum_{i=1}^{k} n_i \Delta D_i^1(p_j) \geqslant 0 \quad \forall j = 0, \ldots, 4N-4, \ \forall k = 1, \ldots, T.$$

It should be shown in Appendix that this is necessary for f to dominate g for all $v_i \in V_1$ and satisfying (H1).

Proposition 2.2 (Atkinson and Bourguignon 1987)**.** *A necessary and sufficient condition for a distribution f to first-degree dominate g for all v_i belonging to V_1 and satisfying (H1) is that (C_1^*) hold.*

3 Second-degree Dominance for points

The favourable permutation of order 2 we introduce below corresponds to the approach of Atkinson and Bourguignon (1987), it is inspired from Moyes (2012, p. 1364).

[5] The question of whether it matters against who a team obtains points has consequences for the definition of what a favourable permutation is. If it matters, a favourable permutation must be limited to cases where the change in points is made through a modification of outcomes of the games between the two teams into consideration. Hence, situations are limited to cases where the teams have either 2 or 4 as the difference in points levels. Otherwise, it does not matter against who a team obtains points. This kind of anonymity principle is consistent with a consequentialist view of the attractiveness of a league. Moreover, this consequentialism is consistent with the fact that the attractiveness functions are defined over the set of *final* distributions of points. The definition 2.1 is valid when we consider 1 as the difference of two teams into consideration. Let us consider that a team of type ℓ obtains one additional point than another team of type i, at the end of the contest. Both teams made two ties against each other. The team of type ℓ made a tie against another team whereas the team of type i lost against the same other team, whatever its type and the points it obtains at the end. The attractiveness of the league increases as the result of a favourable permutation if a league where the team of type ℓ loses against the other team and the team of type i makes a tie against the same other team is more attractive, *ceteris paribus*, than the league described above.

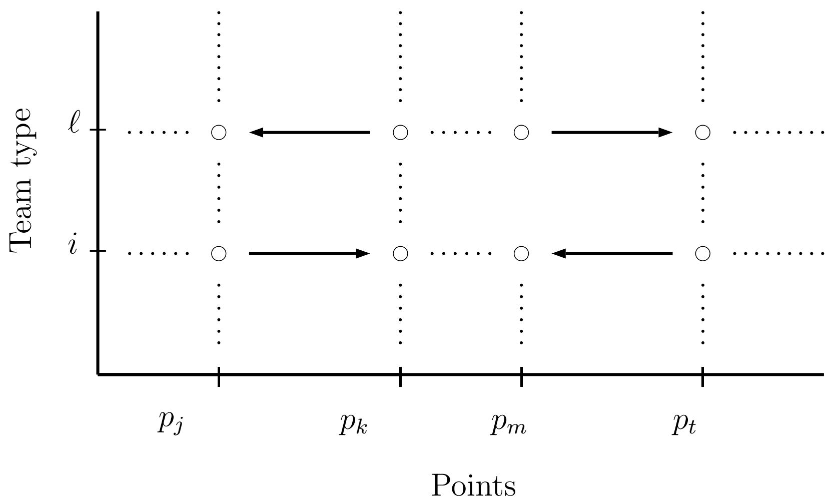
Definition 3.1. *Given two configurations f and g belonging to Ω. We say that f is obtained from g by means of a favourable permutation of order 2 if there exist two types $i \in \{1, \ldots, T-1\}$ and $\ell \in \{2, \ldots, T\}$ so that $i < \ell$, and four point levels $p_k \in \{0, \ldots, 4N-6\}$, $p_{k+\delta} \in \{1, \ldots, 4N-5\}$ so that $p_j < p_k$ and $p_m \in \{1, \ldots, 4N-5\}$, $p_{m+\delta} \in \{2, \ldots, 4N-4\}$ so that $p_m < p_t$; $p_k < p_m$, such that:*

$$f^i(p_k) = g^i(p_k) - \frac{1}{n_i} \ ; \ f^i(p_{k+\delta}) = g^i(p_{k+\delta}) + \frac{1}{n_i} \ ; \ f^i(p_m) = g^i(p_m) + \frac{1}{n_i} \ ; \ f^i(p_{m+\delta}) = g^i(p_{m+\delta}) - \frac{1}{n_i} \ ;$$

$$f^\ell(p_k) = g^\ell(p_k) + \frac{1}{n_\ell} \ ; \ f^\ell(p_{k+\delta}) = g^\ell(p_{k+\delta}) - \frac{1}{n_\ell} \ ; \ f^\ell(p_m) = g^\ell(p_m) - \frac{1}{n_\ell} \ ; \ f^\ell(p_{m+\delta}) = g^\ell(p_{m+\delta}) + \frac{1}{n_\ell} \ ;$$

f and g are identical everywhere else. Equivalently we say that g is obtained from f by means of an unfavourable permutation of order 2.

As an illustration, a favourable permutation of order 2 may be:



The figure clearly shows that a favourable permutation of order 2 combines an unfavourbale permutation and a favourable permutation of the same magnitude, the former involving teams with higher point levels than the latter. The idea is that the positive effect on attractiveness of the favourable permutation more than offsets the negative impact of the unfavourable permutation.

The figure also suggests another interpretation: a favourable permutation of order 2 can be decomposed into (i) a regressive point transfer from the team with p_k to that with p_m which both are of type ℓ and (ii) a progressive point transfer from the team with p_t to that with p_j which both are of type i. The idea is that the positive effect on attractiveness of the progressive point transfer between more popular teams more than offsets the negative impact of the regressive point transfer between less popular teams, at given levels of point.

Suppose now that we are willing to assume $\epsilon_i(p_{j+1}) - \epsilon_i(p_j)$ is non-positive:

(H2) $$\epsilon_i(p_{j+1}) - \epsilon_i(p_j) \geqslant 0 \quad \forall j = 0, \ldots, 4N-6 \text{ and } i = 1, \ldots, T.$$

The assumption (H2) means that the differences in the social marginal valuation of points between groups become lower as we move to higher point levels.[6] Roughly, the assumption may also be interpreted in terms of the degree of diminishing marginal valuation of points $(-\Delta_2 v_i)$ as i rises.[7]

Proposition 3.1. *The attractiveness of a league does not decrease as the result of a favourable permutation of order* 2 *if, and only if,* $v_i \in V_2$ *and (H2) hold.*

The condition (C_1^*) is a first-degree condition, if we are willing to impose further restrictions, then this condition can be weakened. Applying the lemma of summation formula to (3), we have:

$$\Delta A_f = -\sum_{i=1}^{T}\sum_{j=0}^{4N-6} n_i \Delta D_i^2(p_j)\Delta_2 v_i(p_j) + \sum_{i=1}^{T} n_i \Delta D_i^2(p_{4N-5})\Delta_1 v_i(p_{4N-5}) \tag{5}$$

in which $\Delta D_i^2(p_j) = \sum_{k=0}^{j} \Delta D_i^1(p_k) = -\sum_{k=0}^{j} \Delta F^i(p_k)$ for $j = 0, \ldots, 4N-5$ and $i = 1, \ldots, T$.

If (H1) holds, the second term of (5) may be written:

$$\epsilon_T(p_{4N-5})\sum_{i=1}^{T} n_i \Delta D_i^2(p_{4N-5}) + \epsilon_{T-1}(p_{4N-5})\sum_{i=1}^{T-1} n_i \Delta D_i^2(p_{4N-5}) + \ldots + \epsilon_1(p_{4N-5}) n_1 \Delta D_1^2(p_{4N-5})$$

and a sufficient condition for this term to be non-negative is that:

$$\sum_{i=1}^{k} n_i \Delta D_i^2(p_{4N-5}) \geqslant 0 \quad \forall k, k = 1, \ldots, T. \tag{6}$$

From (H1), we can state that:

$$\Delta_2 v_i(p_{j-1}) = \sum_{k=i}^{T} [\epsilon_k(p_{j+1}) - \epsilon_k(p_j)].$$

[6] This statement becomes explicit when, recalling from (H1) that

$$\epsilon_i(p_j) = \sum_{k=i}^{T} \epsilon_k(p_j) - \sum_{k=i+1}^{T} \epsilon_k(p_j) \quad \forall i = 1, \ldots T-1, \text{ and } j = 0, \ldots, 4N-4.$$

Thus, (H2) may be written

$$\sum_{k=i}^{T} \epsilon_k(p_{j+1}) - \sum_{k=i+1}^{T} \epsilon_k(p_{j+1}) \leqslant \sum_{k=i}^{T} \epsilon_k(p_j) - \sum_{k=i+1}^{T} \epsilon_k(p_j) \quad \forall i = 1, \ldots T-1, \text{ and } j = 0, \ldots, 4N-4. \tag{4}$$

[7] Formally, from (H1), (4) becomes

$$\Delta_1 v_i(p_{j+1}) - v_i^{(1)}(p_j) \leqslant \Delta_1 v_{i+1}(p_{j+1}) - \Delta_1 v_{i+1}(p_j) \; \forall i = 1, \ldots T-1, \text{ and } j = 0, \ldots, 4N-6.$$
$$\iff -\Delta_2 v_i(p_j) \geqslant -\Delta_2 v_{i+1}(p_j) \; \forall i = 1, \ldots T-1, \text{ and } j = 0, \ldots, 4N-6$$

which makes more explicit the latter interpretation of (H2).

Thus, the first term of (5) may be written:

$$-\sum_{j=0}^{4N-6}\left\{\begin{array}{l}[\epsilon_T(p_{j+1})-\epsilon_T(p_j)]\sum_{i=1}^{T} n_i\Delta D_i^2(p_j)+[\epsilon_{T-1}(p_{j+1})-\epsilon_{T-1}(p_j)]\sum_{i=1}^{T-1} n_i\Delta D_i^2(p_j)\\ +\ldots+[\epsilon_1(p_{j+1})-\epsilon_1(p_j)]n_1\Delta D_1^2(p_j)\end{array}\right\}. \quad (7)$$

It is then apparent from (6) and (7) that a sufficient condition for $\Delta A_f \geqslant 0$ is that:

$$(C_2^*) \qquad \sum_{i=1}^{k} n_i\Delta D_i^2(p_j) \geqslant 0 \quad \forall j=0,\ldots,4N-5 \text{ and } k=1,\ldots,T.$$

It should be shown in Appendix that this is necessary for f to second-degree downward dominate g for all $v_i \in V_2$ and satisfying (H1) and (H2). If so, we have:

Proposition 3.2. *A necessary and sufficient condition for a distribution f to second-degree downward dominate g for all v_i belonging to V_2 and satisfying (H1) and (H2) is that (C_2^*) hold.*

4 Third-degree Downward Dominance for points

For reference in this section we define a new class of functions as follows:

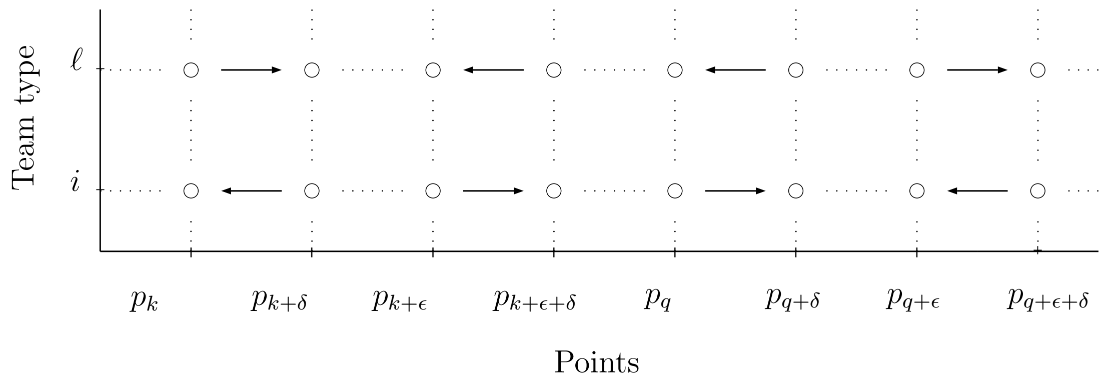
$$V_3=\{v_i\in V_2\ :\ \Delta_3 v_i(p_j)=\Delta_1(\Delta_2 v_i(p_j))<0, \forall j, j=1,\ldots,4N-7; \forall i, i=1,\ldots,T\}.$$

Definition 4.1. *Given two configurations f and g belonging to Ω. We say that f is obtained from g by means of a favourable permutation of order 3 if there exist two types $i\in\{1,\ldots,T-1\}$ and $\ell\in\{2,\ldots,T\}$ so that $i<\ell$, and eight point levels $p_k\in\{0,\ldots,4N-8\}$, $p_{k+\delta}\in\{1,\ldots,4N-7\}$, and $p_{k+\epsilon}\in\{1,\ldots,4N-7\}$, $p_{k+\epsilon+\delta}\in\{2,\ldots,4N-6\}$, and $p_q\in\{2,\ldots,4N-6\}$, $p_{q+\delta}\in\{3,\ldots,4N-5\}$, and $p_{q+\epsilon}\in\{3,\ldots,4N-5\}$, $p_{q+\epsilon+\delta}\in\{4,\ldots,4N-4\}$ with $\delta>0$, $\epsilon\geqslant\delta$, and $p_k<p_q$, such that:*

$$f^i(p_k)=g^i(p_k)+\frac{1}{n_i}\ ;\ f^i(p_{k+\delta})=g^i(p_{k+\delta})-\frac{1}{n_i}\ ;\ f^i(p_{k+\epsilon})=g^i(p_{k+\epsilon})-\frac{1}{n_i}\ ;$$
$$f^i(p_{k+\epsilon+\delta})=g^i(p_{k+\epsilon+\delta})+\frac{1}{n_i}\ ;\ f^i(p_q)=g^i(p_q)-\frac{1}{n_i}\ ;\ f^i(p_{q+\delta})=g^i(p_{q+\delta})+\frac{1}{n_i}\ ;$$
$$f^i(p_{q+\epsilon})=g^i(p_{q+\epsilon})+\frac{1}{n_i}\ ;\ f^i(p_{q+\epsilon+\delta})=g^i(p_{q+\epsilon+\delta})-\frac{1}{n_i}\ ;\ f^\ell(p_k)=g^\ell(p_k)-\frac{1}{n_\ell}\ ;$$
$$f^\ell(p_{k+\delta})=g^\ell(p_{k+\delta})+\frac{1}{n_\ell}\ ;\ f^\ell(p_{k+\epsilon})=g^\ell(p_{k+\epsilon})+\frac{1}{n_\ell}\ ;\ f^\ell(p_{k+\epsilon+\delta})=g^\ell(p_{k+\epsilon+\delta})-\frac{1}{n_\ell}\ ;$$
$$f^\ell(p_q)=g^\ell(p_q)+\frac{1}{n_\ell}\ ;\ f^\ell(p_{q+\delta})=g^\ell(p_{q+\delta})-\frac{1}{n_\ell}\ ;\ f^\ell(p_{q+\epsilon})=g^\ell(p_{q+\epsilon})-\frac{1}{n_\ell}\ ;$$
$$f^\ell(p_{q+\epsilon+\delta})=g^\ell(p_{q+\epsilon+\delta})+\frac{1}{n_\ell}\ ;$$

f and g are identical everywhere else. Equivalently we say that g is obtained from f by means of an unfavourable permutation of order 3.

As an illustration, a favourable permutation of order 3 may be:



Suppose now that we are willing to assume $\epsilon_i(p_{j+1}) - 2\epsilon_i(p_j) + \epsilon_i(p_{j-1})$ is non-positive:

(H3) $$\epsilon_i(p_{j+1}) - 2\epsilon_i(p_j) + \epsilon_i(p_{j-1}) \leqslant 0 \quad \forall j = 1, \ldots, 4N-6 \text{ and } i = 1, \ldots, T.$$

The assumption (H3) implies that $\epsilon_i(p_{j+1}) - \epsilon_i(p_j) \geqslant \epsilon_i(p_j) - \epsilon_i(p_{j-1})$, which means that a favourable permutation of order 2 more than offsets the effect of a unfavourable permutation of order 2 among teams with lower point levels. Moreover, the assumption (H3) implies that $\Delta_3 v_i(p_{j-1}) \leqslant \Delta_3 v_{i+1}(p_{j-1})$, which means that an UNFavourable Composite Transfer (UNFACT) is more valuable for more popular teams than for less popular teams.

Proposition 4.1. *The attractiveness of a league does not decrease as the result of a favourable permutation of order* 3 *if, and only if, $v_i \in V_3$ and (H3) hold.*

Proof. See in Appendix. □

The condition (C_2^*) is a second-degree condition, if we are willing to impose further restrictions, then this condition can be weakened to fulfill the line of thought of Gayant and Le Pape (2017). For that purpose, we introduce the following lemma:

Lemma 4.1. *Downward summation formula. Let $b_1, \ldots, b_N, c_1, \ldots, c_N$ be real numbers. Set $C_j = \sum_{k=j}^{N} c_k$. Then:*

$$\sum_{j=1}^{N} b_j c_j = \sum_{j=2}^{N} C_j (b_j - b_{j-1}) + C_1 b_1.$$

Proof. See in Appendix. □

Applying the lemma of downward summation formula to (5), we have:

$$\Delta A_f = -\sum_{i=1}^{T}\sum_{j=1}^{4N-6} n_i \Delta D_i^3(p_j)\Delta_3 v_i(p_{j-1}) - \sum_{i=1}^{T} n_i \Delta D_i^3(p_0)\Delta_2 v_i(p_0) + \sum_{i=1}^{T} n_i \Delta D_i^2(p_{4N-5})\Delta_1 v_i(p_{4N-5}) \tag{8}$$

in which

$$\Delta D_i^3(p_j) = \sum_{k=j}^{4N-6} \Delta D_i^2(p_k)[p_j - p_{j-1}] \text{ for } j = 0, \dots, 4N-6 \text{ and } i = 1, \dots, T.$$

As stated in Section 4, a sufficient condition for the third term of (8) to be non-negative is described in (6).

If (H1) and (H2) hold, the second term of (8) may be written:

$$\begin{aligned} &-[\epsilon_T(p_1) - \epsilon_T(p_0)]\sum_{i=1}^{T} n_i \Delta D_i^3(p_0) - [\epsilon_{T-1}(p_1) - \epsilon_{T-1}(p_0)]\sum_{i=1}^{T-1} n_i \Delta D_i^3(p_0) \\ &- \dots - [\epsilon_1(p_1) - \epsilon_1(p_0)] n_1 \Delta D_1^3(p_0) \end{aligned}$$

and a sufficient condition for this term to be non-negative is that:

$$\sum_{i=1}^{k} n_i \Delta D_i^3(p_0) \geqslant 0 \quad \forall k, k = 1, \dots, T. \tag{9}$$

From (H1) and (H2), we can state that:

$$\begin{aligned} \Delta_3 v_i(p_{j-1}) &= \Delta_3 v_i(p_j) - \Delta_3 v_i(p_{j-1}) \\ &= \sum_{k=i}^{T}[\epsilon_k(p_{j+1}) - \epsilon_k(p_j)] - \sum_{k=i}^{T}[\epsilon_k(p_j) - \epsilon_k(p_{j-1})] \\ &= \sum_{k=i}^{T}[\epsilon_k(p_{j+1}) - 2\epsilon_k(p_j) + \epsilon_k(p_{j-1})]. \end{aligned} \tag{10}$$

Thus, the first term of (8) may be written:

$$-\sum_{j=1}^{4N-6}\left\{\begin{aligned} &[\epsilon_T(p_{j+1}) - 2\epsilon_T(p_j) + \epsilon_T(p_{j-1})]\sum_{i=1}^{T} n_i \Delta D_i^3(p_j) \\ &+ [\epsilon_{T-1}(p_{j+1}) - 2\epsilon_{T-1}(p_j) + \epsilon_{T-1}(p_{j-1})]\sum_{i=1}^{T-1} n_i \Delta D_i^3(p_j) \\ &+ \dots + [\epsilon_1(p_{j+1}) - 2\epsilon_1(p_j) + \epsilon_1(p_{j-1})] n_1 \Delta D_1^3(p_j) \end{aligned}\right\}. \tag{11}$$

It is then apparent from (6), (9) and (11) that a sufficient condition for $\Delta A_f \geqslant 0$ is that:

$$(C_3^*) \quad \sum_{i=1}^{k} n_i \Delta D_i^3(p_j) \geqslant 0 \text{ and } \sum_{i=1}^{k} n_i \Delta D_i^2(p_{4N-5}) \geqslant 0 \;\; \forall j = 1, \dots, 4N-6 \text{ and } k = 1, \dots, T.$$

It should be shown in Appendix that this is necessary for f to third-degree downward dominate g for all $v_i \in V_3$ and satisfying (H1), (H2) and (H3). If so, we have:

Proposition 4.2. *A necessary and sufficient condition for a distribution f to third-degree downward dominate g for all v_i belonging to V_3 and satisfying (H1), (H2) and (H3) is that (C_3^*) hold.*

This proposition is quite different to all propositions in Atkinson and Bourguignon (1987). It is also different from Lambert and Ramos (2002).

First, there is a difference between third-order sequential dominance on continuous variables and that on grids. According to Fishburn and Lavalle (1995, Corollary 3), if $4N-4 \geqslant 3$, there exists one $v_i \in V_3$ such that no function u_i – three-times differentiable so that $u_i \in V_2$ and its third derivative is negative – having v_i as its restriction in $\mathscr{P}$. In this sense, the criterion we propose in (C_3^*) implies third-degree downward dominance on continuous variables but the converse is not true.

5 Conclusion

The first judgment we propose requires teams with larger fan-bases to have a higher social valuation of points, at any given point level. Hence, it is judged more attractive to observe a victory of a team with larger fan-base against a team with smaller fan-base than either the converse or a tie between the two teams, *ceteris paribus*. The second judgment we present means that the differences in the social marginal valuation of points between teams with different fan-bases become lower as we move to higher point levels. Hence, it is judged more attractive to observe a victory of a team with larger fan-base against a team with smaller fan-base at lower point levels than at higher point levels. This judgment may also be interpreted in terms of diminishing marginal valuation of points as the fan-base decreases. According to this interpretation, a league is more attractive if it is more balanced among teams with larger fan-bases. The third judgment we endorse is the exact opposite of that exposed by Lambert and Ramos (2002). A league is judged even more attractive if it is more balanced among teams with larger fan-bases at higher point levels. The dominance criterion we introduce is characterized by the the fulfillment of the three judgments, and it ranks leagues in line with all indices which satisfy such judgments.

However, our criteria do not allow for comparing leagues with different marginal distributions of fan-bases. This limitation rules out the possibility to make international comparisons of leagues. One future possibility is to use joint distribution functions – as Moyes (2012) does – in order to compare configurations in which both the distribution of point and that of fan-bases are different. Another future possibility is to introduce open leagues in the framework so that promotions and relegations can happen (Gayant and Le Pape 2017).

Appendix

Proof of Proposition 2.1

Proof. [Sufficency] From definition 2.1, let f and g belong to Ω so that f is obtained from g by means of a favourable permutation of order 1. Taking recourse of the expression in (3), suppose that:

$$\Delta A_f = \sum_{j=0}^{4N-5} \sum_{t=1}^{T} n_t \Delta_1 v_t(p_j) \Delta D_t^1(p_j) \geqslant 0$$

in which $\Delta D_t^1(p_j) = -F_t(p_j) = -\sum_{k=0}^{j} \left(f^t(p_k) - g^t(p_k)\right)$. Thus,

$$\begin{aligned}
\Delta A_f \geqslant 0 \iff & n_i \Delta_1 v_i(p_k) \Delta D_i^1(p_k) + n_\ell \Delta_1 v_\ell(p_k) \Delta D_\ell^1(p_k) \geqslant 0 \\
\iff & n_i \Delta_1 v_i(p_k) \times \frac{1}{n_i} + n_\ell \Delta_1 v_\ell(p_k) \times \frac{1}{n_\ell} \geqslant 0 \\
\iff & \Delta_1 v_i(p_k) - \Delta_1 v_\ell(p_k) \geqslant 0,
\end{aligned} \tag{12}$$

which is stated by (H1).

[Necessity] If either $v_i \in V_1$ or (H1) is not fulfilled, then it is possible to find two distributions f and g belonging to Ω so that f is obtained from g by means of a favourable permutation of order 1 and $\Delta A_f < 0$. Suppose for example that (H1) is violated so that there exist $p_j \in \{0, \ldots, 4N-5\}$ and $i \in \{1, \ldots, T-1\}$ such that $v_i(p_k) > v_{i+1}(p_k)$ and $v_i(p_{k+1}) < v_{i+1}(p_{k+1})$. From definition 2.1, consider then $\ell = i+1$ and $p_{k+\delta} = p_{k+1}$. We get:

$$\begin{aligned}
\Delta A_f =& v_i(p_{k+1}) - v_i(p_k) - v_{i+1}(p_{k+1}) + v_{i+1}(p_k) \\
=& v_i(p_{k+1}) - v_{i+1}(p_{k+1}) - \left[v_i(p_k) - v_{i+1}(p_k)\right].
\end{aligned}$$

Then $\Delta A_f < 0$.

Suppose for example that $v_i \notin V_1$ so that there exist $p_j \in \{0, \ldots, 4N-5\}$ and $i \in \{1, \ldots, T-1\}$ such that $\Delta_1 v_i(p_k) < 0$ and $\Delta_1 v_{i+1}(p_k) > 0$. From definition 2.1, consider then $\ell = i+1$ and $p_{k+\delta} = p_{k+1}$. We get:

$$\Delta A_f = \Delta_1 v_i(p_k) - \Delta_1 v_{i+1}(p_k).$$

Then $\Delta A_f < 0$. $\square$

Proof of Proposition 3.1

Proof. [Sufficency] From definition 3.1, let f and g belong to Ω so that f is obtained from g by means of a favourable permutation of order 2. Taking recourse of the expression in (3), suppose that:

$$\begin{aligned}
&\Delta A_f \geqslant 0 \\
\iff &n_i \Delta_1 v_i(p_k) \Delta D_i^1(p_k) + n_i \Delta_1 v_i(p_m) \Delta D_i^1(p_m) \\
&+ n_\ell \Delta_1 v_\ell(p_k) \Delta D_\ell^1(p_k) + n_i \Delta_1 v_i(p_m) \Delta D_\ell^1(p_m) \geqslant 0 \\
\iff &\Delta_1 v_i(p_k) - \Delta_1 v_i(p_m) - \Delta_1 v_\ell(p_k) \\
&+ \Delta_1 v_\ell(p_m) \geqslant 0
\end{aligned}$$

Let us consider $p_m = p_{k+1}$, it turns out that:

$$\Delta A_f = \Delta_2 v_\ell(p_k) - \Delta_2 v_i(p_k) \geqslant 0,$$

which is ensured by (H2).
[Necessity] If either $v_i \in V_2$ or (H2) is not fulfilled, then it is possible to state that the attractiveness of a league decreases as the result of a favourable permutation of order 2.
Case 1: Suppose for example that (H2) is violated so that there exist $p_j \in \{0, \ldots, 4N-6\}$ and $i \in \{1, \ldots, T-1\}$ such that $\Delta_2 v_{i+1}(p_k) < \Delta_2 v_i(p_k)$. From definition 3.1, consider then $\ell = i+1$ and $p_{k+\delta} = p_{k+1}$. We get:

$$\begin{aligned}
&\Delta_2 v_\ell(p_k) < \Delta_2 v_i(p_k) \\
\iff &\Delta_1 v_\ell(p_{k+1}) - \Delta_1 v_\ell(p_k) < \Delta_1 v_i(p_{k+1}) - \Delta_1 v_i(p_k).
\end{aligned}$$

Let $p_{k+1} = p_m$, we have

$$\begin{aligned}
&\Delta_1 v_\ell(p_m) - \Delta_1 v_\ell(p_k) < \Delta_1 v_i(p_m) - \Delta_1 v_i(p_k) \\
\iff &(p_{k+1} - p_k) \Delta_1 v_\ell(p_m) - \Delta_1 v_\ell(p_k) - \Delta_1 v_i(p_m) + \Delta_1 v_i(p_k) < 0.
\end{aligned}$$

We obtain $\Delta A_f < 0$.
Case 2: Suppose for example that $v_i \notin V_2$ so that there exist $p_j \in \{0, \ldots, 4N-6\}$ and $i \in \{1, \ldots, T-1\}$ such that $\Delta_2 v_{i+1}(p_k) < 0$ and $\Delta_2 v_i(p_k) > 0$. Then, $\Delta_2 v_{i+1}(p_k) < \Delta_2 v_i(p_k)$ and the proof is analogue to the case 1. □

Proof of Proposition 4.1

Proof. [Sufficency] From definition 4.1, let f and g belong to Ω so that f is obtained from g by means of a favourable permutation of order 3. Taking recourse of the expression in (3), suppose that:

$$
\begin{aligned}
& \Delta A_f \geqslant 0 \\
\iff & n_i \Delta_1 v_i(p_k) \Delta D_i^1(p_k) + n_i \Delta_1 v_i(p_{k+\epsilon}) \Delta D_i^1(p_{k+\epsilon}) \\
& + n_i \Delta_1 v_i(p_q) \Delta D_i^1(p_q) + n_i \Delta_1 v_i(p_{q+\epsilon}) \Delta D_i^1(p_{q+\epsilon}) \\
& + n_\ell \Delta_1 v_\ell(p_k) \Delta D_\ell^1(p_k) + n_\ell \Delta_1 v_\ell(p_{k+\epsilon}) \Delta D_\ell^1(p_{k+\epsilon}) \\
& + n_\ell \Delta_1 v_\ell(p_q) \Delta D_\ell^1(p_q) + n_\ell \Delta_1 v_\ell(p_{q+\epsilon}) \Delta D_\ell^1(p_{q+\epsilon}) \geqslant 0 \\
\iff & -\Delta_1 v_i(p_k) + \Delta_1 v_i(p_{k+\epsilon}) \\
& + \Delta_1 v_i(p_q) - \Delta_1 v_i(p_{q+\epsilon}) \\
& + \Delta_1 v_\ell(p_k) - \Delta_1 v_\ell(p_{k+\epsilon}) \\
& - \Delta_1 v_\ell(p_q) + \Delta_1 v_\ell(p_{q+\epsilon}) \geqslant 0
\end{aligned}
$$

Let us consider $\epsilon = 1$, it turns out that:

$$\Delta A_f = \Delta_2 v_i(p_k) - \Delta_2 v_i(p_q) - \Delta_2 v_\ell(p_k) + \Delta_2 v_\ell(p_q) \geqslant 0,$$

Setting that $q = k + 1$, we have:

$$\Delta_3 v_\ell(p_k) \geqslant \Delta_3 v_i(p_k)$$

which is ensured by (H3).
[Necessity] If either $v_i \in V_3$ or (H3) is not fulfilled, then it is possible to state that the attractiveness of a league decreases as the result of a favourable permutation of order 3.
Case 1: Suppose for example that (H3) is violated so that there exist $p_j \in \{0, \ldots, 4N-7\}$ and $i \in \{1, \ldots, T-1\}$ such that $\Delta_3 v_{i+1}(p_k) < \Delta_3 v_i(p_k)$. From definition 4.1, consider then $\ell = i + 1$. We get:

$$
\begin{aligned}
& \Delta_3 v_\ell(p_k) < \Delta_3 v_i(p_k) \\
\iff & \Delta_2 v_\ell(p_{k+1}) - \Delta_2 v_\ell(p_k) < \Delta_2 v_i(p_{k+1}) - \Delta_2 v_i(p_k).
\end{aligned}
$$

Let $p_{k+1} = p_q$, we have

$$
\begin{aligned}
& \Delta_2 v_\ell(p_q) - \Delta_2 v_\ell(p_k) < \Delta_2 v_i(p_q) - \Delta_1 v_i(p_k) \\
\iff & \Delta_1 v_\ell(p_{q+1}) - \Delta_1 v_\ell(p_q) - \Delta_1 v_\ell(p_{k+1}) + \Delta_1 v_\ell(p_k) \\
& < \Delta_1 v_i(p_{q+1}) - \Delta_1 v_i(p_q) - \Delta_1 v_i(p_{k+1}) + \Delta_1 v_i(p_k)
\end{aligned}
$$

Let $p_{k+1} = p_{k+\epsilon}$ and $p_{q+1} = p_{q+\epsilon}$, we have

$$\Delta_1 v_\ell(p_{q+\epsilon}) - \Delta_1 v_\ell(p_q) - \Delta_1 v_\ell(p_{k+\epsilon}) + \Delta_1 v_\ell(p_k) < \Delta_1 v_i(p_{q+\epsilon}) - \Delta_1 v_i(p_q) - \Delta_1 v_i(p_{k+\epsilon}) + \Delta_1 v_i(p_k).$$

We obtain $\Delta A_f < 0$.
Case 2: Suppose for example that $v_i \notin V_3$ so that there exist $p_j \in \{0, \dots, 4N-7\}$ and $i \in \{1, \dots, T-1\}$ such that $\Delta_3 v_{i+1}(p_k) < 0$ and $\Delta_3 v_i(p_k) > 0$. Then, $\Delta_3 v_{i+1}(p_k) < \Delta_3 v_i(p_k)$ and the proof is analogue to the case 1. $\square$

Necessity of (C_1^*)

Proof. Suppose that $\Delta A_f \geqslant 0$, that is:

$$\sum_{j=0}^{4N-5} \sum_{i=1}^{T} n_i \Delta_1 v_i(p_j) \Delta D_i^1(p_j) \geqslant 0 \ \forall v_i \in V_1.$$

If there exists $k \in \{0, \dots, 4N-5\}$ such that for at least one $i \in \{1, \dots, T\}$, $\Delta D_i^1(p_k) < 0$, and $\Delta D_i^1(p_k) \leqslant 0$ for all $i \in \{1, \dots, T\}$, then consider v_i for which $v_i(p_j) = v_i(p_{j+1})$ for all $j \neq k$ and $v_i(p_k) < v_i(p_{k+1})$. We have $v_i \in V_1$ and:

$$\Delta A_f = \sum_{i=1}^{T} n_i \Delta_1 v_i(p_k) \Delta D_i^1(p_k).$$

Since $p_{k+1} > p_k$, we have $\Delta A < 0$, which contradicts the starting assumption. $\square$

Necessity of (C_2^*)

Proof. Suppose that $\Delta A_f \geqslant 0$, that is:

$$-\sum_{j=0}^{4N-6} \sum_{i=1}^{T} n_i \Delta_2 v_i(p_j) \Delta D^2 v_i(p_j) + \sum_{i=1}^{T} n_i \Delta_1 v_i(p_{4N-5}) \Delta D_i^2(p_{4N-5}) \geqslant 0 \ \forall v_i \in V_2.$$

If there exists $k \in \{0, \dots, 4N-6\}$ such that for at least one $i \in \{1, \dots, T\}$, $\Delta D_i^2(p_k) < 0$, and $\Delta D_i^2(p_k) \leqslant 0$ for all $i \in \{1, \dots, T\}$, then consider v_i for which $v_i(p_j) = v_i(p_{j+1})$ for all $j \geqslant k+1$, $v_i(p_k + 1) > v_i(p_k)$ and $\Delta_1 v_i(p_j) = \Delta_1 v_i(p_{j+1})$ for all $j \neq k$. We have $v_i \in V_2$ and:

$$\Delta A_f = -\sum_{i=1}^{T} n_i \Delta D_i^2(p_k) \Delta_2 v_i(p_k).$$

We have $\Delta A < 0$, which contradicts the starting assumption. $\square$

Downward summation formula

Proof. Let $b_1, \dots, b_N, c_1, \dots, c_N$ be real numbers. Set $C_j = \sum_{k=j}^{N} c_k$. Then for every $j > 0$,

$$c_j = \sum_{k=j}^{N} c_k - \sum_{k=j+1}^{N} c_k = C_j - C_{j+1}.$$

It turns out that:

$$\begin{aligned}
\sum_{j=1}^{N} b_j c_j &= \sum_{j=1}^{N-1} b_j (C_j - C_{j+1}) + b_N c_N \\
&= \sum_{j=1}^{N-1} b_j C_j - \sum_{j=1}^{N-1} b_j C_{j+1} + b_N c_N = \sum_{j=2}^{N-1} b_j C_j + b_1 C_1 - \sum_{j=1}^{N-2} b_j C_{j+1} - b_{N-1} C_N + b_N c_N \\
&= \sum_{j=2}^{N-1} (b_j - b_{j-1}) C_j + b_1 C_1 - b_{N-1} C_N + b_N c_N \\
&= \sum_{j=2}^{N-1} (b_j - b_{j-1}) C_j + (b_N - b_{N-1}) C_N - (b_N - b_{N-1}) C_N + b_1 C_1 - b_{N-1} C_N + b_N c_N \\
&= \sum_{j=2}^{N} (b_j - b_{j-1}) C_j - b_N C_N + b_{N-1} C_N + b_1 C_1 - b_{N-1} C_N + b_N c_N
\end{aligned}$$

By definition $C_N = c_N$, thus:

$$\sum_{j=1}^{N} b_j c_j = \sum_{j=2}^{N} (b_j - b_{j-1}) C_j + b_1 C_1$$

which concludes the proof. □

Necessity of (C_3^*)

Proof. Suppose that $\Delta A_f \geqslant 0$, that is:

$$-\sum_{j=1}^{4N-6} \sum_{i=1}^{T} n_i \Delta_3 v_i(p_j) \Delta D^3 v_i(p_j) - \sum_{i=1}^{T} \Delta D_i^3(p_0) \Delta_2 v_i(p_0) + \sum_{i=1}^{T} n_i \Delta_1 v_i(p_{4N-5}) \Delta D_i^2(p_{4N-5}) \geqslant 0 \; \forall v_i \in V_3.$$

If there exists $k \in \{1, \ldots, 4N-6\}$ such that for at least one $i \in \{1, \ldots, T\}$, $\Delta D_i^3(p_k) < 0$, and $\Delta D_i^3(p_k) \leqslant 0$ for all $i \in \{1, \ldots, T\}$, then consider v_i for which $v_i(p_j) = v_i(p_{j+1})$ for all $j \geqslant k+1$, $v_i(p_k + 1) > v_i(p_k)$ and $\Delta_1 v_i(p_j) = \Delta_1 v_i(p_{j+1})$ for all $j \leqslant k$, $\Delta_1 v_i(p_k) > \Delta_1 v_i(p_{k-1})$, and $\Delta_2 v_i(p_j) = \Delta_2 v_i(p_{j-1})$ for all $j \neq k$. We have $v_i \in V_3$ and:

$$\Delta A_f = -\sum_{i=1}^{T} n_i \Delta D_i^3(p_k) \Delta_3 v_i(p_{k-1}).$$

We have $\Delta A < 0$, which contradicts the starting assumption. □

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