Computing isogeny classes of typical principally polarized abelian surfaces over the rationals

Jean Kieffer (Harvard)

COUNT – Computations and their uses in number theory, CIRM Luminy
February 28, 2023

Joint work with Raymond van Bommel, Shiva Chidambaram, and Edgar Costa (MIT)

Isogenies

Fix a base field k, a number field.

Definition

An isogeny between two abelian varieties is $\varphi : A \twoheadrightarrow B$ such that $\# \ker \varphi < \infty$.

Isogenies are obtained by taking quotients by finite rationals subgroups. Being isogenous is an equivalence relation, as we have $\varphi^\vee : B^\vee \to A^\vee$.

We are interested in the isogeny class of A over k.

Isogeny classes

Two abelian varieties in the same isogeny class share many properties, including

- L-function
- Mordell–Weil rank
- Endomorphism algebra $\mathrm{End}(A) \otimes \mathbb{Q}$.

Theorem (Faltings)

The isogeny class of A over k is finite.

Can construct (finite, connected) isogeny graphs:

- Vertices are abelian varieties in an isogeny class,
- Edges are irreducible isogenies, e.g. labeled by degree.

Question

What are the possible isogeny graphs?

Elliptic curves over the rationals: the LMFDB

We can explore isogeny graphs of elliptic curves over $\mathbb{Q}$ at `www.LMFDB.org`.

- Ignoring degrees, we find 10 different graphs:

Size	1	2	3	4	6	8
Examples	37.a	26.b	11.a	27.a, 20.a, 17.a	14.a, 21.a	15.a, 30.a

- All edge labels, i.e. degrees of irreducible isogenies, are prime.
- Not all primes ℓ appear as isogeny degrees: only

$$\ell \in \{2, \ldots, 19, 37, 43, 67, 163\}.$$

Elliptic curves over the rationals: theorems

Lemma

Any isogeny $\varphi : E \to E'$ can be factored as $E \xrightarrow{[n]} E \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} E_n = E'$, where $\deg(\varphi_i) = \ell_i$ are primes.

Theorem (Mazur)

If $\varphi : E \to E'$ defined over $\mathbb{Q}$ has prime degree ℓ, then $\ell \in \{2, \ldots, 19, 37, 43, 67, 163\}$.

Theorem (Kenku)

Any isogeny class of elliptic curves over $\mathbb{Q}$ has size at most 8.

Chiloyan – Lozano-Robledo 2021

Complete classification of possible labeled isogeny graphs.

The LMFDB contains examples for all of these graphs.

Higher dimensions?

No such complete picture away from elliptic curves over $\mathbb{Q}$.

One approach is to collect data:

Algorithmic problem

Given an abelian variety A over a number field k, compute its isogeny class.

Eventually restrict to the simplest higher-dimensional case:

- Abelian surfaces
- endowed with principal polarizations
- over $k = \mathbb{Q}$
- that are typical, i.e. $\mathrm{End}(A^{\mathrm{al}}) = \mathbb{Z}$.

These are all Jacobians of genus 2 curves over $\mathbb{Q}$.

www.LMFDB.org contains genus 2 curves with small discriminants, grouped by (heuristic) isogeny class of their Jacobians, but these isogeny classes are not complete.

Algorithmic approach

Algorithmic problem

Given an abelian variety A over a number field k, compute its isogeny class.

For an elliptic curve $E/\mathbb{Q}$:

1. Search for ℓ-isogenies $E \to E'$ for each ℓ in Mazur's list. This is a finite problem.
2. Reapply on E' as needed.

In general:

1. Reduce to finitely many isogeny types. (E.g., "prime degree" for elliptic curves)
2. Compute a finite number of possible degrees. We now face a finite problem.
3. Search for all isogenies of a given type and degree.
4. Reapply as needed.

Classification of isogenies

$\varphi : A \to B$ isogeny between principally polarized abelian varieties.

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \wr\downarrow \lambda_A & & \wr\downarrow \lambda_B \\ A^\vee & \xleftarrow[\varphi^\vee]{} & B^\vee \end{array} \quad \rightsquigarrow \quad \mu = \lambda_A^{-1} \circ \varphi^\vee \circ \lambda_B \circ \varphi \in \mathrm{End}(A).$$

Recall that $\mathrm{End}(A)$ has a positive Rosati involution $\dagger$ defined by $\mu^\dagger = \lambda_A^{-1} \circ \mu^\vee \circ \lambda_A$.

Theorem (Mumford)

There is a bijection

$$\left\{\varphi : A \to B\right\} \longleftrightarrow \left\{(\mu, K) \; : \; \begin{array}{l} \mu \in \mathrm{End}(A)^\dagger, \;\; \mu > 0 \\ K \subseteq A[\mu] \;\; \text{maximal isotropic} \end{array}\right\}$$

$$\varphi \longmapsto \left(\lambda_A^{-1} \circ \varphi^\vee \circ \lambda_B \circ \varphi, \, \ker \varphi\right).$$

Irreducible isogeny types

Assume now that $\mathrm{End}(A)^\dagger = \mathbb{Z}$. (True in particular if A is typical).

Any $\varphi : A \to B$ satisfies: $\ker(\varphi)$ is maximal isotropic in $A[n]$ for some $n \in \mathbb{Z}_{\geq 1}$.

Up to decomposing φ, can assume $n = \ell^e$ is a prime power.

Lemma

Assume $e \geq 3$. If $K \subset A[\ell^e]$ is maximal isotropic, then $\ell K \cap A[\ell^{e-2}]$ is maximal isotropic in $A[\ell^{e-2}]$.

Thus, any isogeny $\varphi : A \to B$ can always be factored as

$$A = A_0 \xrightarrow{\varphi_1} A_1 \xrightarrow{\varphi_2} A_2 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_n} A_n = B,$$

where $\ker(\varphi_i)$ is maximal isotropic in $A_{i-1}[\ell_i]$ or $A_{i-1}[\ell_i^2]$, for ℓ_i prime.

Irreducible isogeny types for abelian surfaces

Further assume that A is an abelian surface (with p.p., and $\mathrm{End}(A)^\dagger = \mathbb{Z}$). Then the other p.p. abelian surfaces in the isogeny class of A can be enumerated by looking at isogenies φ of the following types:

1. 1-step: $K := \ker(\varphi)$ is a maximal isotropic subgroup of $A[\ell]$, so $K \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$,
2. 2-step: K is a maximal isotropic subgroup of $A[\ell^2]$ and $K \simeq (\mathbb{Z}/\ell\mathbb{Z})^2 \times \mathbb{Z}/\ell^2\mathbb{Z}$.

Degree ℓ^2 and ℓ^4 respectively.

Over $\mathbb{Q}^{\mathrm{al}}$, every 2-step isogeny decomposes as a sequence of two 1-step isogenies, in $\ell + 1$ different ways (permuted by Galois).

Computing isogeny classes

Algorithmic problem

Given a p.p. abelian variety A over a number field k, compute its isogeny class.

	Elliptic curves $/\mathbb{Q}$	Typical p.p. abelian surfaces $/\mathbb{Q}$
Isogeny types	Prime degree	1-step or 2-step ✓
Possible degrees	Mazur's theorem	?
Search for isogenies		

Serre's open image theorem

Theorem (Mazur)

If $\varphi : E \to E'$ defined over $\mathbb{Q}$ has prime degree ℓ, then $\ell \in \{2, \ldots, 19, 37, 43, 67, 163\}$.

No uniform result à la Mazur is known for abelian surfaces. However:

Serre's open image theorem

If A is a typical abelian surface, then its Galois representation has open image in $\mathrm{GSp}_4(\widehat{\mathbb{Z}})$. Thus, $A[\ell]$ has nontrivial rational subgroups only for finitely many ℓ's.

Includes all primes for which 1-step and 2-step isogenies exist. Results of Lombardo, Zywina give bounds on such ℓ's (depending on A), but are impractical.

Dieulefait's algorithm

Results of Lombardo, Zywina give bounds on ℓ as in Serre's open image theorem (depending on A), but are impractical.

Instead we use:

Algorithm (Dieulefait)[1]

Input: Conductor of A and a finite list of L-polynomials

Output: Finite superset of primes ℓ with reducible mod-ℓ Galois representation.

Example where the only possibilities are isogenies of degree 31^2:

$$C\colon y^2 + (x+1)y = x^5 + 23x^4 - 48x^3 + 85x^2 - 69x + 45.$$

[1] See also Banwait–Brumer–Kim–Klagsbrun–Mayle–Srinivasan–Vogt (2023).

Computing isogeny classes

Algorithmic problem

Given a p.p. abelian variety A over a number field k, compute its isogeny class.

	Elliptic curves $/\mathbb{Q}$	Typical p.p. abelian surfaces $/\mathbb{Q}$
Isogeny types	Prime degree	1-step or 2-step ✓
Possible degrees	Mazur's theorem	Dieulefait's algorithm ✓
Search for isogenies	?	??

Modular polynomials

Elliptic curves: usually search for ℓ-isogenies using algebraic equations for the cover of modular curves $X_0(\ell) \to X(1)$.

E.g., the modular polynomials $\Phi_\ell(x, y) \in \mathbb{Z}[x, y]$ defined by

$$\Phi_\ell(j, j') = 0 \iff \exists\, \varphi : E_j \longrightarrow E_{j'} \text{ such that } \ker \varphi \simeq \mathbb{Z}/\ell\mathbb{Z}.$$

Size grows as $\widetilde{O}(\ell^3)$, big but manageable (28MB for $\ell = 163$).

Abelian surfaces: Modular polynomials for p.p. abelian surfaces are impractical.
More variables: $\Phi_\ell(x_1, x_2, x_3, y) \in \mathbb{Q}(x_1, x_2, x_3)[y]$.
Size grows as $\widetilde{O}(\ell^{15})$ (K. 2022), already $\gg 29\,\mathrm{GB}$ for $\ell = 7$.

We use complex-analytic methods instead.

Moduli space of elliptic curves

Let $E/\mathbb{C}$ be an elliptic curve. Moduli space: $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}_1$.

Can choose $\tau \in \mathbb{H}_1$ and an equation $E : y^2 = x^3 - 27c_4x - 54c_6$ such that

$$\begin{aligned} E(\mathbb{C}) &\simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \\ \frac{dx}{2y} &\mapsto 2\pi i\, dz. \end{aligned}$$

Then c_4, c_6 are modular forms:

$$c_4 = E_4(\tau), \quad c_6 = E_6(\tau), \quad \text{hence} \quad j(E) = j(\tau) = 1728\frac{E_4(\tau)}{E_4(\tau)^3 - E_6(\tau)^2}.$$

Theorem

The graded $\mathbb{C}$-algebra of modular forms on $\mathbb{H}_1$ for $\mathrm{SL}_2(\mathbb{Z})$ is $\mathbb{C}[E_4, E_6]$.

Moreover E_4, E_6 have integral, primitive Fourier expansions.

Hence c_4, c_6 are indeed “the right invariants” to consider.

Moduli space of p.p. abelian surfaces

A complex p.p. abelian surface takes the form $\mathbb{C}^2/(\mathbb{Z}^2 + \tau\mathbb{Z}^2)$ with $\tau \in \mathbb{H}_2$.
Moduli space: $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2$.

Theorem (Igusa)

The graded $\mathbb{C}$-algebra of (scalar-valued) Siegel modular forms of even weight on $\mathbb{H}_2$ for $\mathrm{Sp}_4(\mathbb{Z})$ is $\mathbb{C}[M_4, M_6, M_{10}, M_{12}]$, where the M_i are algebraically independent.

Normalized such that the M_j have primitive, integral Fourier expansions and M_{10}, M_{12} are cusp forms.
Explicit relations with the Igusa–Clebsch invariants I_2, I_4, I_6, I_{10} of a genus 2 curve:

$$M_4 = 2^{-2} I_4, \qquad M_6 = 2^{-3}(I_2 I_4 - 3 I_6),$$
$$M_{10} = -2^{-12} I_{10}, \qquad M_{12} = 2^{-15} I_2 I_{10}.$$

The M_j's are "the right invariants" on the moduli space of p.p. abelian surfaces.

Analytic isogenies

Enumerating isogenous abelian varieties is easy on the complex-analytic side.

- **Elliptic curves:** the complex tori ℓ-isogenous to $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ are given by

  $$\mathbb{C}/(\mathbb{Z}+\tfrac{1}{\ell}\eta\tau\mathbb{Z})$$

  where $\eta \in \mathrm{SL}_2(\mathbb{Z})$ are coset representatives for $\Gamma^0(\ell)\backslash\mathrm{SL}_2(\mathbb{Z})$.
  Note: $\frac{1}{\ell}\eta\tau = \gamma\tau$ where $\gamma = \left(\begin{smallmatrix}1 & 0\\ 0 & \ell\end{smallmatrix}\right)\eta \in \mathrm{GL}_2(\mathbb{Q})^+$.

- **Abelian surfaces:** explicit sets $S_1(\ell)$, $S_2(\ell) \subset \mathrm{GSp}_4(\mathbb{Q})^+$ such that for $i = 1, 2$,

  $$\left\{\text{ab. surfaces } i\text{-step } \ell\text{-isogenous to } \mathbb{C}^2/(\mathbb{Z}^2+\tau\mathbb{Z}^2)\right\} = \left\{\mathbb{C}^2/\left(\mathbb{Z}^2+\gamma\tau\mathbb{Z}^2\right)\right\}_{\gamma\in S_i(\ell)}.$$

  Cf. explicit formulas for Hecke operators $T(\ell)$, $T_1(\ell^2)$.

Algorithmic problem

Decide when $\gamma\tau \in \mathbb{H}_2$ is attached to an abelian surface defined over $\mathbb{Q}$.

Construction of algebraic integers

Theorem (corollary of Igusa)

If f is a Siegel modular form of even weight k with integral Fourier coefficients, then $12^k f \in \mathbb{Z}[M_4, M_6, M_{10}, M_{12}]$.

Theorem

Let $\tau \in \mathbb{H}_2$ such that there exists $\lambda \in \mathbb{C}^\times$ with $\lambda^j M_j(\tau) \in \mathbb{Z}$ for $j \in \{4, 6, 10, 12\}$.
If f is a Siegel modular form of even weight k with integral Fourier coefficients, then

$$\prod_{\gamma \in S_i(\ell)} \Big(X - \big(12\lambda\ell^3 \det(c_\gamma\tau + d_\gamma)^{-1}\big)^k f(\gamma\tau) \Big) \in \mathbb{Z}[X].$$

Thus, for each $j \in \{4, 6, 10, 12\}$, the complex numbers

$$N(j, \gamma) := \big(12\lambda\ell^3 \det(c_\gamma\tau + d_\gamma)^{-1}\big)^j M_j(\gamma\tau) \qquad \text{for } \gamma \in S_i(\ell),\ i = 1 \text{ or } 2,$$

form a Galois-stable set of algebraic integers.

Algorithm and certification

Input: Invariants $m_4, m_6, m_{10}, m_{12} \in \mathbb{Z}$ of a genus 2 curve, a prime ℓ, and $i \in \{1, 2\}$.

Output: Invariants of all i-step ℓ-isogenous abelian surfaces.

1. Compute complex balls that provably contain:
   - $\tau \in \mathbb{H}_2$
   - $\lambda \in \mathbb{C}^\times$ such that $\lambda^j M_j(\tau) = m_j$ for $j \in \{4, 6, 10, 12\}$
   - $N(j, \gamma)$, for each $j \in \{4, 6, 10, 12\}$ and $\gamma \in S_i(\ell)$.
2. Keep the γ_0's such that $N(j, \gamma_0)$ contains an integer m'_j for each $j \in \{4, 6, 10, 12\}$. The m'_j are putative invariants for the abelian surface attached to $\gamma_0 \tau$.
3. Confirm that $N(j, \gamma_0) = m'_j$ by certifying the vanishing of

$$\prod_{\gamma \in S_i(\ell)} \left(N(j, \gamma) - m'_j \right) \in \mathbb{Z}.$$

   We need to recompute $N(j, \gamma_0)$ (only!) to a much higher precision.

Example, continued

Let $\ell = 31$, $i = 1$ and

$$C\colon y^2 + (x+1)y = x^5 + 23x^4 - 48x^3 + 85x^2 - 69x + 45.$$

Working at 300 bits of precision, there is only one γ_0 such that the $N(j, \gamma_0)$ for $j \in \{4, 6, 10, 12\}$ contain integers:

$$\begin{aligned} N(4,\gamma_0) &= \alpha^2 \cdot 318972640 \pm 7.8 \times 10^{-47}, \\ N(6,\gamma_0) &= \alpha^3 \cdot 1225361851336 \pm 5.5 \times 10^{-39}, \\ N(10,\gamma_0) &= \alpha^5 \cdot 10241530643525839 \pm 1.6 \times 10^{-29}, \\ N(12,\gamma_0) &= -\alpha^6 \cdot 307105165233242232724 \pm 4.6 \times 10^{-22} \end{aligned}$$

where $\alpha = 2^2 \cdot 3^2 \cdot 31$.

We certify these equalities by working at 4 128 800 bits of precision. Use certified quasi-linear time algorithms for the evaluation of modular forms (K. 2022).*

Reconstructing a genus 2 curve

Given $(m'_4, m'_6, m'_{10}, m'_{12}) = (318972640, 1225361851336, 10241530643525839, \ldots)$, find a corresponding curve C' such that $\mathrm{Jac}(C)$ and $\mathrm{Jac}(C')$ are isogenous over $\mathbb{Q}$.

Mestre's algorithm yields

$$y^2 = -1624248x^6+5412412x^5-6032781x^4+876836x^3-1229044x^2-5289572x-1087304,$$

a quadratic twist by -83761 of the desired curve

$$C' : y^2+xy = -x^5+2573x^4+92187x^3+2161654285x^2+406259311249x+9395128975286 2.$$

We reapply the algorithm to C', and we only find the original curve.

Comments:

- 113 minutes of CPU time for this example; 90% is to certify the results.
- Can independently create a certificate for the isogeny (6.5 hours and 3 MB).

LMFDB data

Originally 63 107 typical genus 2 curves in 62 600 isogeny classes.

By computing isogeny classes, we found 21 923 new curves.

Size	1	2	3	4	5	6	7	8	9	10	12	16	18
Count	51 549	2 672	6 936	420	756	164	40	45	3	2	3	9	1

LMFDB data

Originally 63 107 typical genus 2 curves in 62 600 isogeny classes.

By computing isogeny classes, we found 21 923 new curves.

Size	1	2	3	4	5	6	7	8	9	10	12	16	18
Count	51 549	2 672	6 936	420	756	164	40	45	3	2	3	9	1

Observation

A 2-step 2-isogeny (of degree 16) always implies an existence of a second one.
This explains the 6913 $\triangle$ and the 756 $\bowtie$ we found.

The whole computation took 75 hours. Only 3 classes took more than 10 minutes:

- 349.a: 56 min, isogeny of degree 13^4.
- 353.a: 23 min, isogeny of degree 11^4.
- 976.a: 19 min, checking that no isogeny of degree 29^4 exists.

Upcoming to LMFDB

A new set of 5 235 806 curves due to Sutherland is soon to be added to the LMFDB. Of these, 1 823 592 are typical, split amongst 1 538 149$\pm\varepsilon$ isogeny classes.

We found 688 094 new curves (in 97 days). Counts per size:

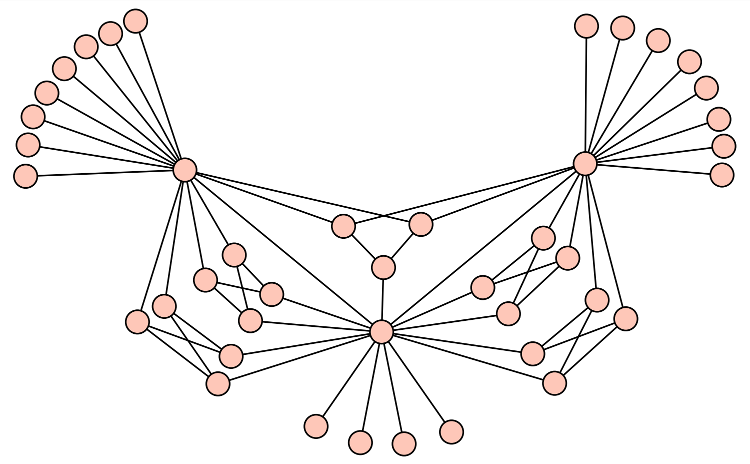
1	2	3	4	5	6	7	8	≥ 9
1 098 812	125 694	212 000	58 310	16 925	15 459	498	6 073	4 270

We discovered irreducible isogenies of degree

$$2^2 \text{ (= Richelot isogenies)}, \; 2^4, 3^2, 3^4, 5^2, 5^4, 7^2, 7^4, 11^4, 13^2, 13^4, 17^2, 31^2.$$

- Size 2: 75% have degree 2^2, 22% have degree 3^4, and then 3^2, 5^4, 5^2, 7^4, 7^2, ...
- Size 3: 99.2% are $\triangle$ of degree 2^4 isogenies.
- Size 4: 97.8% are $\succ\!\!-$ of Richelot isogenies.
- Size 5: 99.8% are $\bowtie$ of degree 2^4 isogenies.
- Size 6: 75% + 15% are two graphs consisting of Richelot isogenies.

Isogeny graph consisting of 42 Richelot isogenous curves outside our database, with conductor $497051100 = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17^2$:



The end

https://arxiv.org/abs/2301.10118

Thank you.