

# An Algorithmic Reduction Theory for Binary Codes: LLL and More

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## This work

Propose *analogues* from lattices to binary codes (Defs, Algs, Bounds).

Speed-up cryptanalytic algorithms for code-based cryptography. ?

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## This talk

- Recall the LLL algorithm for lattices.
- Adapt it to codes.

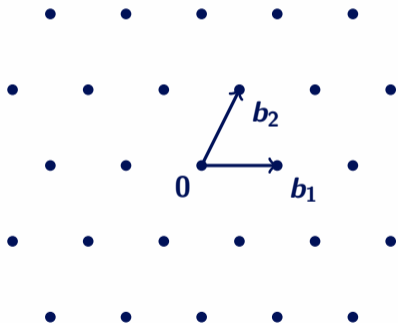
What notion of orthogonality for binary codewords?

# Lattices & Codes

Lattice

$$\mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^k\} \subset \mathbb{R}^n$$

Euclidean

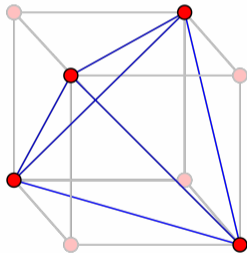


$$\mathcal{L} = b_1\mathbb{Z} + b_2\mathbb{Z}$$

Binary Code

$$\mathcal{C}(B) := \{\sum_i x_i b_i : x \in \mathbb{F}_2^k\} \subset \mathbb{F}_2^n$$

Hamming



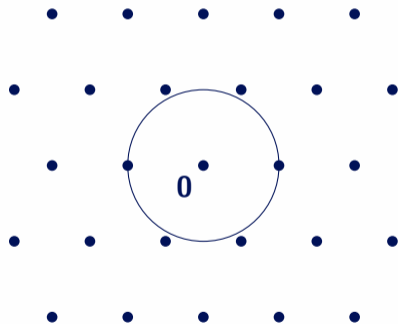
$$\mathcal{C} = \{000, 011, 101, 110\}$$

# Hard Problems

Lattice

$$\lambda_1(\mathcal{L}) := \min_{x \in \mathcal{L} \setminus \{0\}} \|x\|_2$$

Euclidean

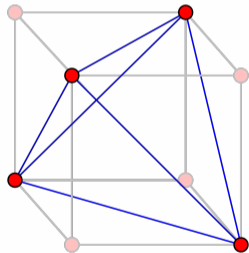


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Binary Code

$$d_{\min}(\mathcal{C}) := \min_{x \in \mathcal{C} \setminus \{0\}} |x|$$

Hamming

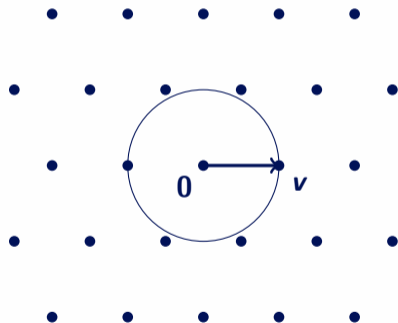


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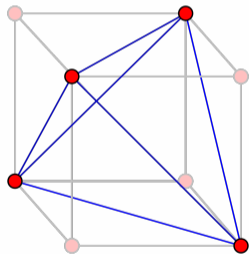
Find a *short nonzero* vector  $v \in \mathcal{L}(B)$ .



$$\mathcal{L} = b_1\mathbb{Z} + b_2\mathbb{Z}$$

## Binary Code

Find a *short nonzero* codeword  $v \in \mathcal{C}(B)$ .

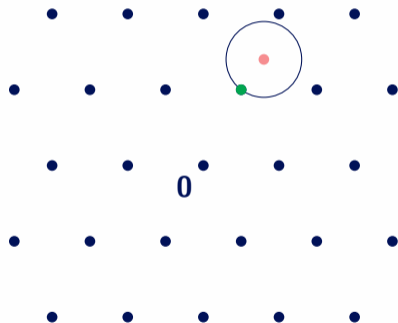


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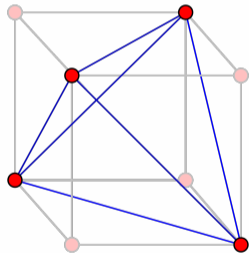
Given a target  $t \in \mathbb{R}^n$  find  
a *close* vector  $v \in \mathcal{L}(B)$ .



$$\mathcal{L} = b_1\mathbb{Z} + b_2\mathbb{Z}$$

## Binary Code

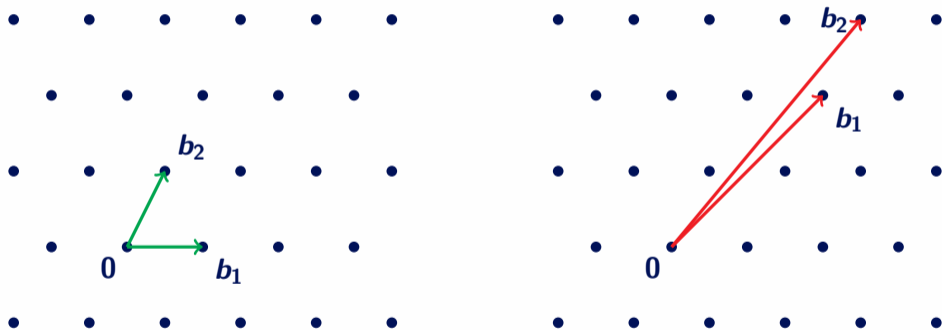
Given a target  $t \in \mathbb{F}_2^n$  find  
a *close* codeword  $c \in \mathcal{C}(B)$ .



$$\mathcal{C} = \{000, 011, 101, 110\}$$

# Basis Reduction

$B' := B \cdot U$  is a basis of  $\mathcal{L}(B)$  if and only if  $U \in \text{GL}_d(\mathbb{Z})$ .

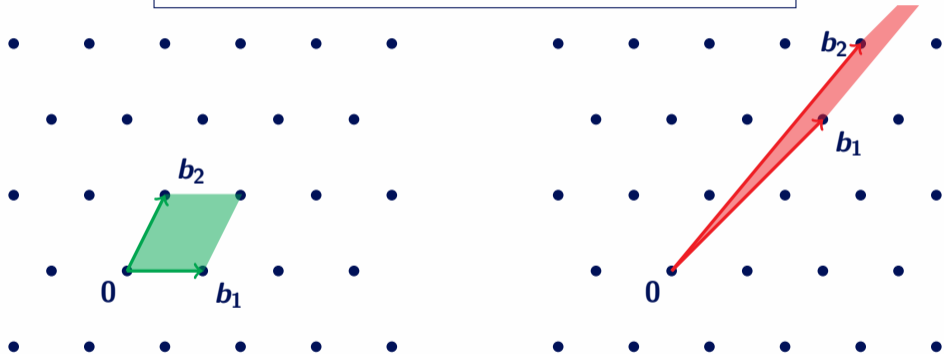




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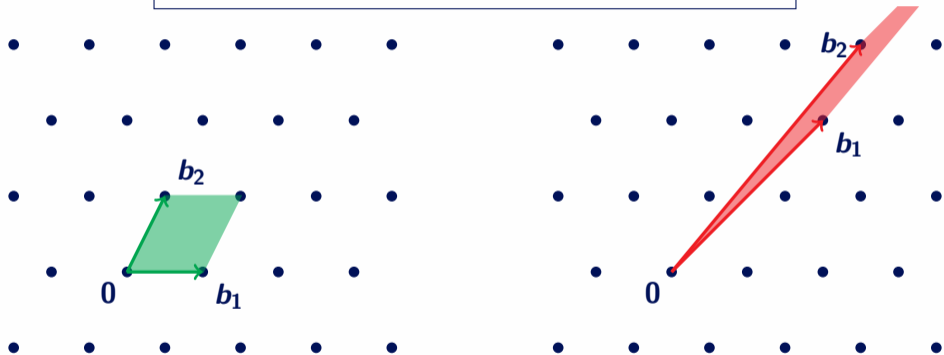
Invariant:  $\det(\mathcal{L}) := \text{Vol}(\mathbb{R}^n / \mathcal{L}) = \det(B)$ .



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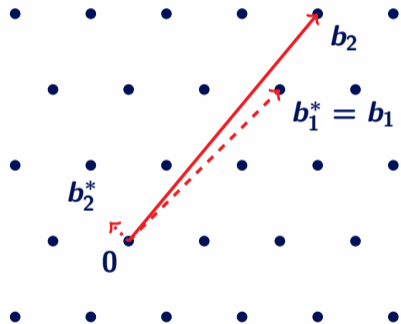
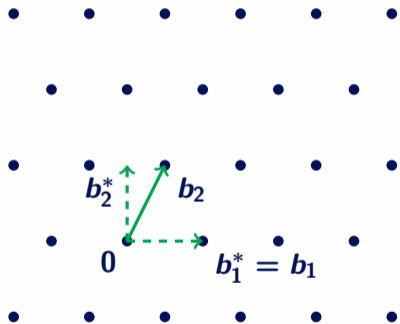


Find a ‘*good*’ lattice basis of  $\mathcal{L}$ .

Short and somewhat orthogonal.

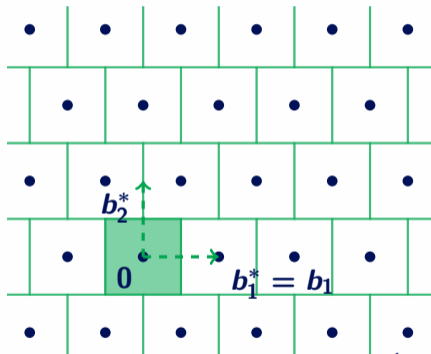
# Gram-Schmidt Orthogonalisation

$$b_i^* := \underbrace{\pi(b_1, \dots, b_{i-1})^\perp}_{\pi_i}(b_i)$$



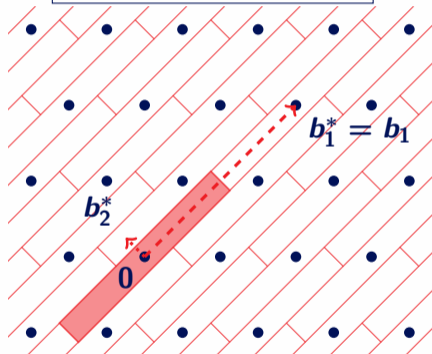
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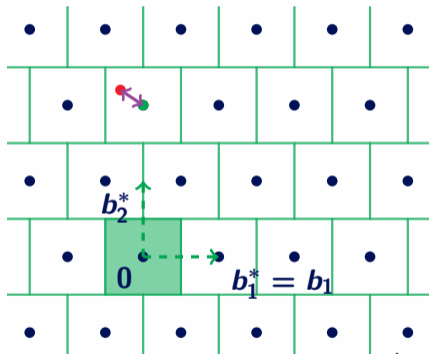
Fundamental Area:  $\mathcal{F}_{B^*} := \prod_{i=1}^k \left[ -\frac{1}{2}b_i^*, \frac{1}{2}b_i^* \right)$

$$\prod_{i=1}^k \|b_i^*\| = \det(\mathcal{L})$$



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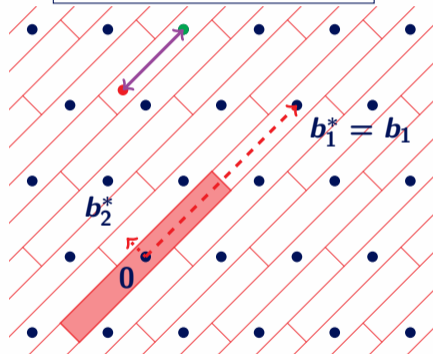
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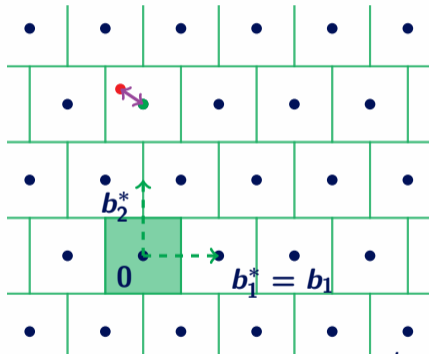
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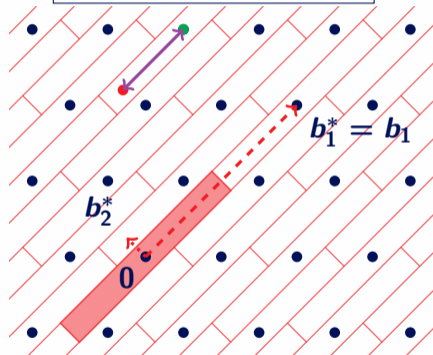
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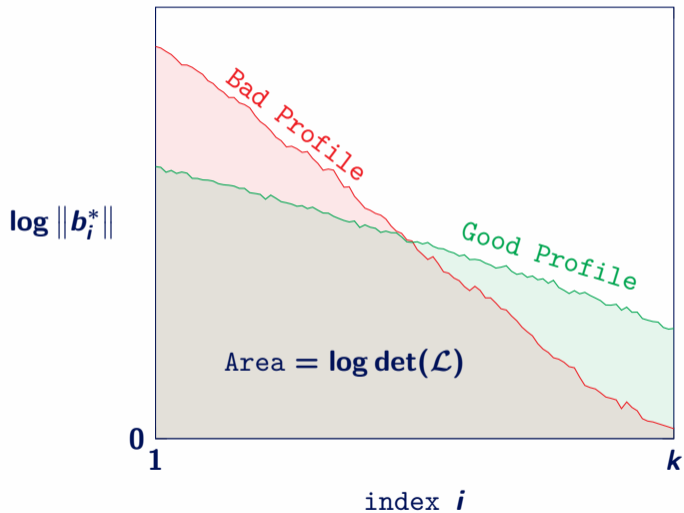
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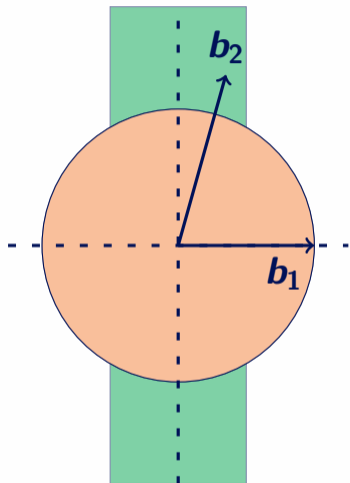


Good Basis:  
 $\|b_1^*\| \approx \dots \approx \|b_k^*\|$

# Profile



# Lagrange Reduction (k=2)



## Wristwatch Lemma

For any lattice  $\mathcal{L}$  of rank 2  
there exists a basis  $(b_1, b_2)$  s.t.

$$\|b_1\| \leq \|b_2\|$$

$$|\langle b_1, b_2 \rangle| \leq \frac{1}{2} \|b_1\|$$



$$\|b_1^*\| \leq \sqrt{\frac{4}{3}} \cdot \|b_2^*\|$$



# LLL Reduction

## Definition

A basis  $\mathbf{B}$  of  $\mathcal{L}$  is LLL-reduced if  $(\pi_i(\mathbf{b}_i), \pi_i(\mathbf{b}_{i+1}))$  is Lagrange Reduced for all  $i < k$ .

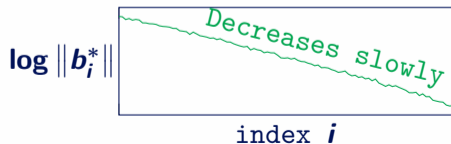
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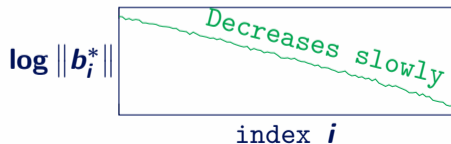
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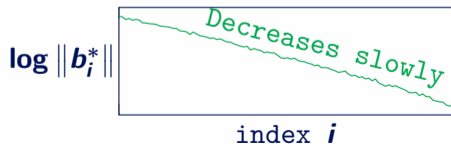
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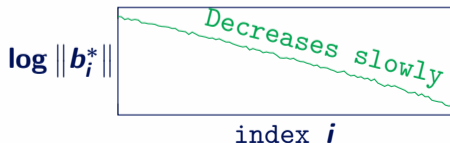
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While  $\exists i$  s.t.  $(\pi_i(\mathbf{b}_i), \pi_i(\mathbf{b}_{i+1}))$  is *not* Lagrange Reduced, Lagrange Reduce it.

Termination in poly-time:

Requires a slight relaxation.  
( $\epsilon$ -Lagrange Reduced)

Proof argument:

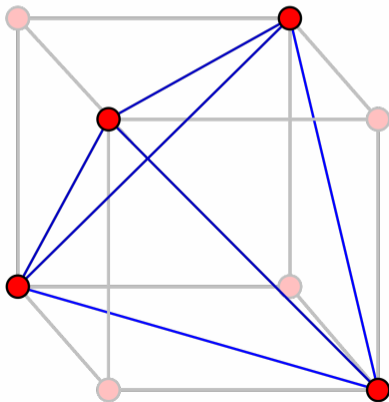
$$P = \sum_{i \leq k} (n + 1 - i) \cdot \log \|\mathbf{b}_i^*\|$$

Decreases by  $\epsilon$  at each step  
and is lower-bounded.

# Binary Codes

$$\mathcal{C}(B) := \{\sum_i x_i b_i : x \in \mathbb{F}_2^k\} \subset \mathbb{F}_2^n$$

$k$ -dimensional subspace of  $\mathbb{F}_2^n$ ,  
endowed with the Hamming metric.



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To mimic LLL we need  
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$|\mathbf{x} \oplus \mathbf{y}| = |\mathbf{x}| + |\mathbf{y}|$   
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$\mathbf{x}$ 

0	0	1	0	1
---	---	---	---	---

 $\text{Supp}(\mathbf{x}) = \{3, 5\}$   
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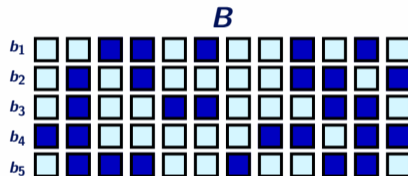
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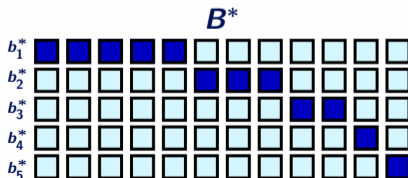
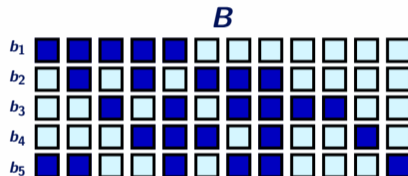
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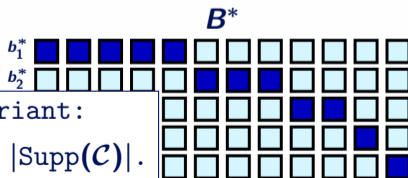
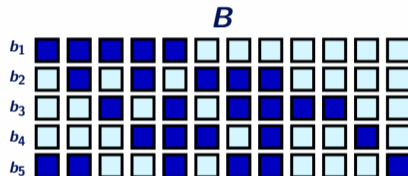
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Invariant:  
 $\sum_{i=1}^k |b_i^*| = |\text{Supp}(C)|$ .

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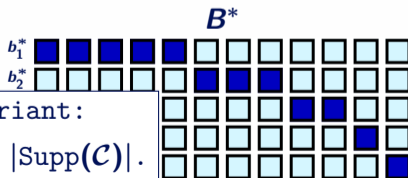
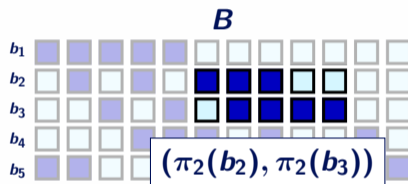
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# Langrange Reduction (for codes)

## Lemma

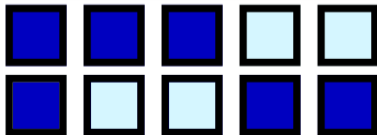
For any code  $\mathcal{C}$  of support size  $n = |\text{Supp}(\mathcal{C})|$ ,  
and rank  $k = 2$ , there exists a basis  $\mathbf{b}_1, \mathbf{b}_2$  s.t.

$$|\mathbf{b}_1| \leq |\mathbf{b}_2|, \quad |\text{Supp}(\mathbf{b}_1) \cap \text{Supp}(\mathbf{b}_2)| \leq \frac{1}{2} \cdot |\mathbf{b}_1|.$$



$$|\mathbf{b}_1^*| \leq 2 \cdot |\mathbf{b}_2^*|$$

(Lattice case:  $\|\mathbf{b}_1^*\| \leq \sqrt{4/3} \|\mathbf{b}_2^*\|$ )





## LLL Reduction (for codes)

Algorithm

While  $\exists i$  s.t.  $(\pi_i(\mathbf{b}_i), \pi_i(\mathbf{b}_{i+1}))$  is *not* Lagrange Reduced,  
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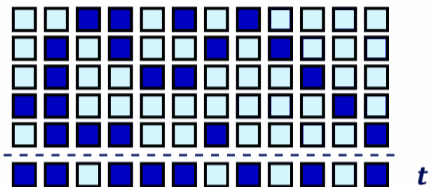


$$|\mathbf{b}_1| - \frac{\lceil \log_2 |\mathbf{b}_1| \rceil}{2} \leq \frac{n-k}{2} + 1$$

Griesmer bound!

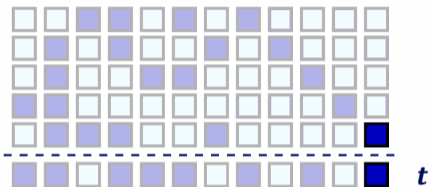
# Babai Decoding (for codes)

Prange Decoding



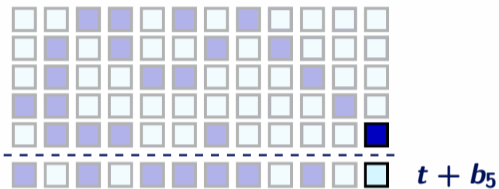
# Babai Decoding (for codes)

Prange Decoding



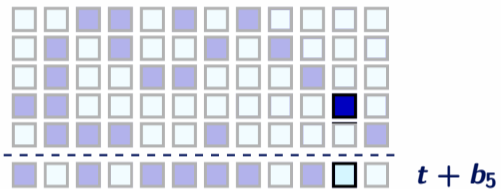
# Babai Decoding (for codes)

Prange Decoding



# Babai Decoding (for codes)

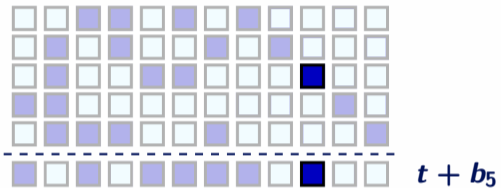
Prange Decoding





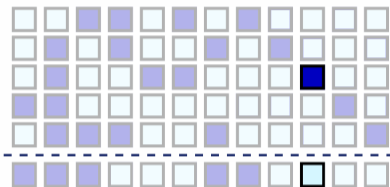
# Babai Decoding (for codes)

Prange Decoding



# Babai Decoding (for codes)

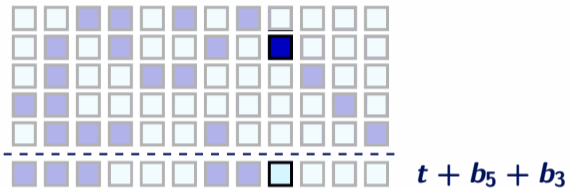
Prange Decoding



$$t + b_5 + b_3$$

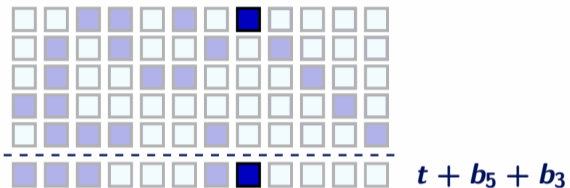
# Babai Decoding (for codes)

Prange Decoding



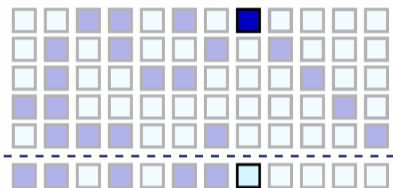
# Babai Decoding (for codes)

Prange Decoding



# Babai Decoding (for codes)

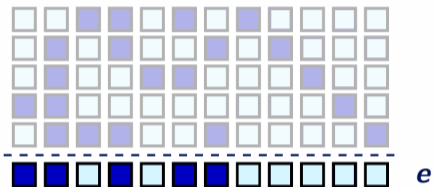
Prange Decoding



$$t + b_5 + b_3 + b_1$$

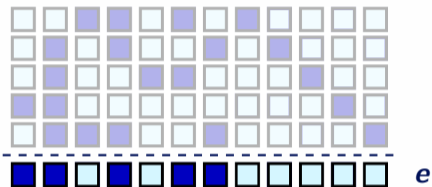
# Babai Decoding (for codes)

Prange Decoding



# Babai Decoding (for codes)

## Prange Decoding



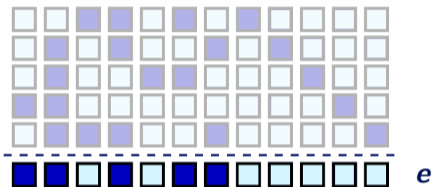
Fun. Domain:  $\mathbf{e} \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{\mathbf{0}\}^k$

Worst-case:  $|\mathbf{e}| \leq n - k$

Average-case:  $\mathbb{E}[|\mathbf{e}|] = \frac{n-k}{2}$

# Babai Decoding (for codes)

## Prange Decoding

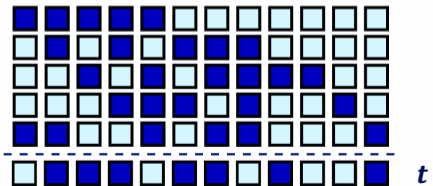


Fun. Domain:  $e \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{0\}^k$

Worst-case:  $|e| \leq n - k$

Average-case:  $\mathbb{E}[|e|] = \frac{n-k}{2}$

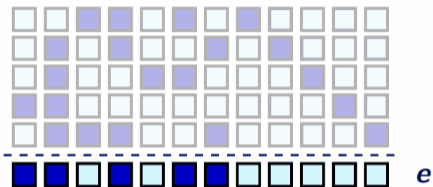
## Babai Decoding





# Babai Decoding (for codes)

## Prange Decoding

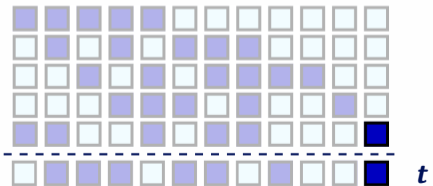


Fun. Domain:  $e \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{0\}^k$

Worst-case:  $|e| \leq n - k$

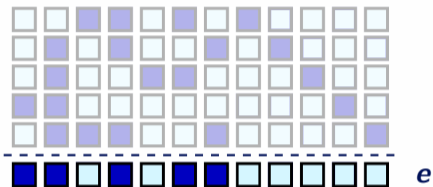
Average-case:  $\mathbb{E}[|e|] = \frac{n-k}{2}$

## Babai Decoding



# Babai Decoding (for codes)

## Prange Decoding

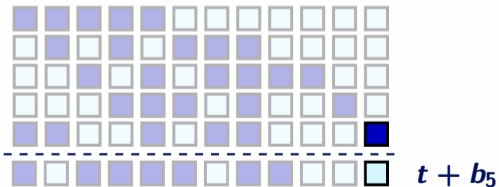


Fun. Domain:  $e \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{0\}^k$

Worst-case:  $|e| \leq n - k$

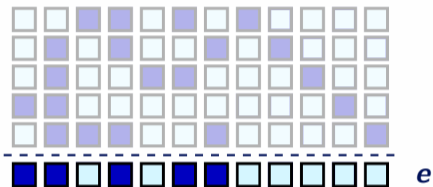
Average-case:  $\mathbb{E}[|e|] = \frac{n-k}{2}$

## Babai Decoding



# Babai Decoding (for codes)

## Prange Decoding

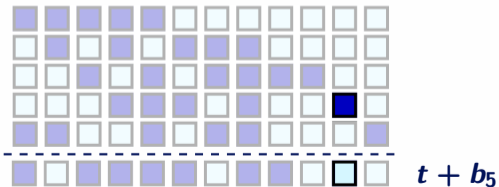


Fun. Domain:  $e \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{0\}^k$

Worst-case:  $|e| \leq n - k$

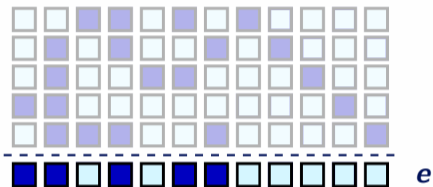
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## Babai Decoding



# Babai Decoding (for codes)

## Prange Decoding

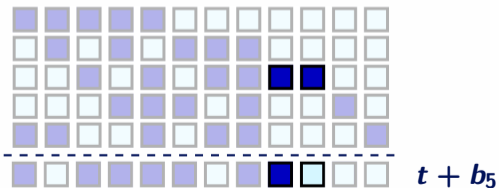


Fun. Domain:  $e \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{0\}^k$

Worst-case:  $|e| \leq n - k$

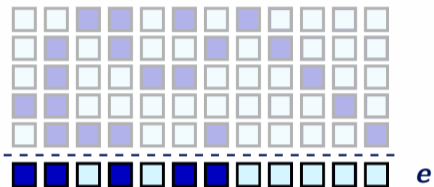
Average-case:  $\mathbb{E}[|e|] = \frac{n-k}{2}$

## Babai Decoding



# Babai Decoding (for codes)

## Prange Decoding

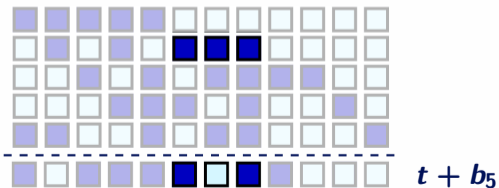


Fun. Domain:  $e \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{0\}^k$

Worst-case:  $|e| \leq n - k$

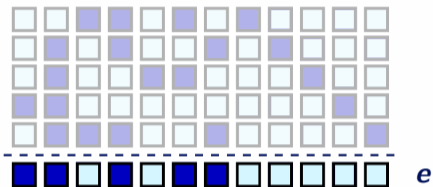
Average-case:  $\mathbb{E}[|e|] = \frac{n-k}{2}$

## Babai Decoding



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## Prange Decoding

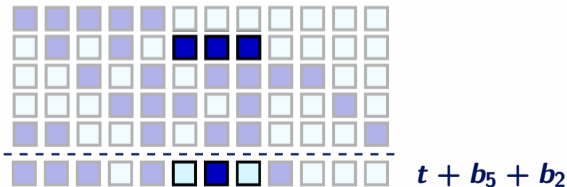


Fun. Domain:  $e \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{0\}^k$

Worst-case:  $|e| \leq n - k$

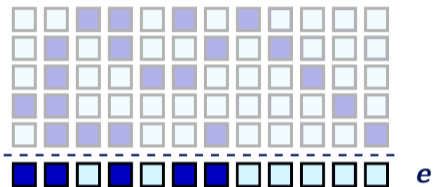
Average-case:  $\mathbb{E}[|e|] = \frac{n-k}{2}$

## Babai Decoding



# Babai Decoding (for codes)

## Prange Decoding

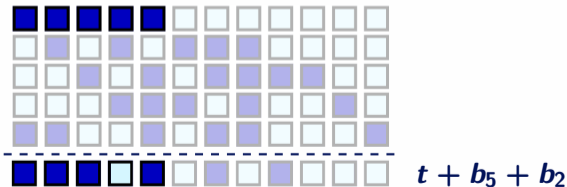


Fun. Domain:  $e \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{0\}^k$

Worst-case:  $|e| \leq n - k$

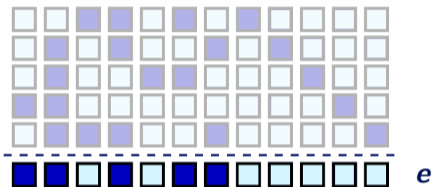
Average-case:  $\mathbb{E}[|e|] = \frac{n-k}{2}$

## Babai Decoding



# Babai Decoding (for codes)

## Prange Decoding

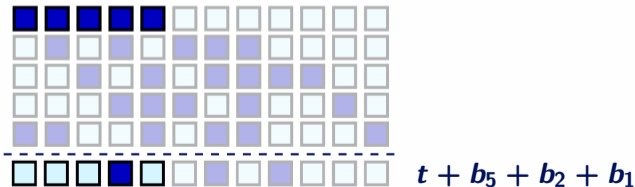


Fun. Domain:  $e \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{0\}^k$

Worst-case:  $|e| \leq n - k$

Average-case:  $\mathbb{E}[|e|] = \frac{n-k}{2}$

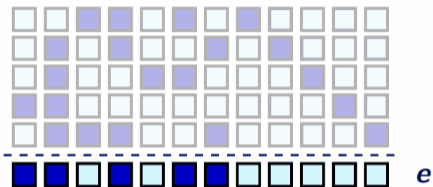
## Babai Decoding





# Babai Decoding (for codes)

## Prange Decoding

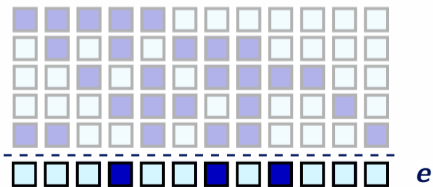


Fun. Domain:  $\mathbf{e} \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{\mathbf{0}\}^k$

Worst-case:  $|\mathbf{e}| \leq n - k$

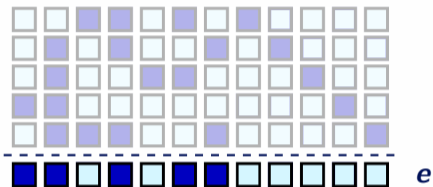
Average-case:  $\mathbb{E}[|\mathbf{e}|] = \frac{n-k}{2}$

## Babai Decoding



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## Prange Decoding

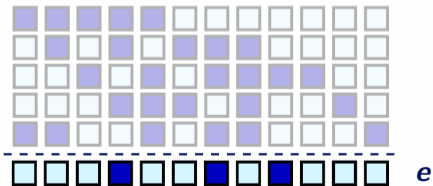


Fun. Domain:  $e \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{0\}^k$

Worst-case:  $|e| \leq n - k$

Average-case:  $\mathbb{E}[|e|] = \frac{n-k}{2}$

## Babai Decoding



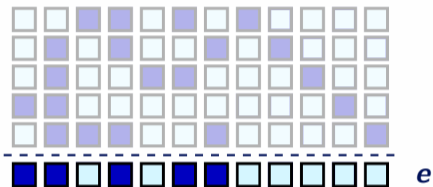
Fun. Domain(\*):  $e \in \mathcal{F}_{B^*} := \prod_{i=1}^k \mathcal{B}_{\lfloor |b_i^*|/2 \rfloor}^{|b_i^*|}$

Worst-case:  $|e| \leq \sum_{i=1}^n \lfloor |b_i^*|/2 \rfloor \ll n - k$

Average-case:  $\mathbb{E}[|e|] \leq \frac{n-k}{2}$

# Babai Decoding (for codes)

## Prange Decoding

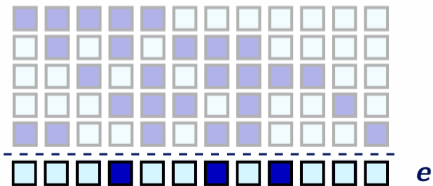


Fun. Domain:  $e \in \mathcal{F} := \mathbb{F}_2^{n-k} \times \{0\}^k$

Worst-case:  $|e| \leq n - k$

Average-case:  $\mathbb{E}[|e|] = \frac{n-k}{2}$

## Babai Decoding

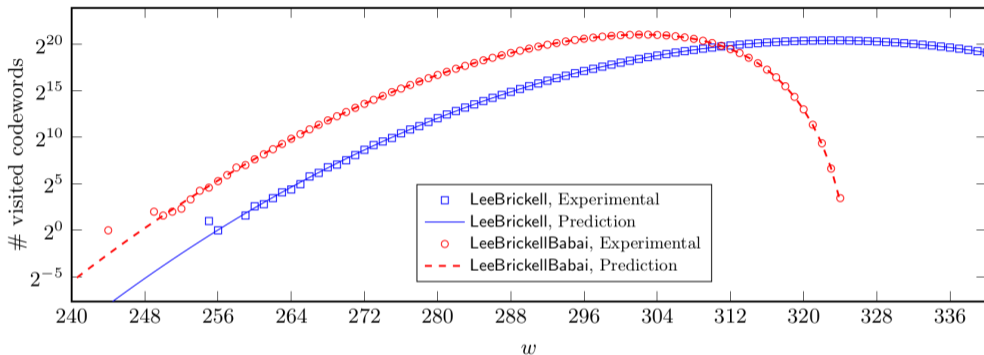


Fun. Domain(\*):  $e \in \mathcal{F}_{B^*} := \prod_{i=1}^k \mathcal{B}_{\lfloor |b_i^*|/2 \rfloor}^{|b_i^*|}$

Worst-case:  $|e| \leq \sum_{i=1}^n \lfloor |b_i^*|/2 \rfloor \ll n - k$   
Better when

Average-case:  $\mathbb{E}[|e|] \leq \frac{n-k}{2}$  more reduced!

# Improved hybrid algorithms



$\Theta(n^{0.717} / \log(n))$  heuristic speed-up over standard Lee Brickell.

Compatible with more advanced algorithms.

# Thank you!

Paper:

[eprint.iacr.org/2020/869](https://eprint.iacr.org/2020/869)

Code & Experiments:

[github.com/lucas/CodeRed](https://github.com/lucas/CodeRed)

*Open-source!*