Questionnaire “Logic and Computability”

Summer Term 2024

Contents

7 Predicate Logic

7.1 Predicates and Quantifiers

7.1.1 Model the following declarative sentences with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

(a) Some students like Alice.
(b) Every teacher likes Bob.
(c) Some students like every teacher.
(d) Some students and Bob play a game.
(e) Not every student plays games.
(f) Some teachers play no games.

7.1.2 Model the following declarative sentences with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

(a) Alice has no sister.
(b) A person who wears a crown is either a king or a queen.
(c) Not everybody likes everybody.
(d) Everybody loves somebody.

7.1.3 Model the following declarative sentences with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

(a) The construction takes a long time, is noisy, and blocks the sun.
(b) If there is no school, at least one parent of each kid has to take vacation and cannot got to work.
(c) All students have to take the exam eventually.

7.1.4 Model the following declarative sentences with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

(a) If all kids wear gloves, then all parents will be happy.
(b) All kids love pizza and spaghetti.
(c) All kids are fun, energetic, and cannot sit still.

7.1.5 Consider the following declarative sentence (known as *Goldbach's Conjecture*):

"Every even integer greater than 2 is equal to the sum of two prime numbers."

Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

7.1.6 Consider the following declarative sentences:

"Every person who has the same parents as John Doe and is different from John Doe himself is a sibling of John Doe."

Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.
Also, model the same sentence in propositional logic, as detailed as possible. Clearly indicate the intended meaning of each propositional variable you use.

7.1.7 Translate the following sentences into predicate logic. Be as precise as possible. Give the meaning of any function and predicate symbols you use.

(a) Nobody knows everybody.

(b) All birds can fly, except for penguins and ostrichs.

(c) Not all birds can fly, but some birds can fly.

(d) All kids are cute and quite if and only if they are sleeping

7.1.8 Translate the following sentence into predicate logic. Be as precise as possible. Give the meaning of any function and predicate symbols you use.

Every even integer greater than 2 is equal to the sum of two prime numbers.

7.1.9 Consider the following declarative sentence:

"For every natural number it holds that it is prime if and only if there is no smaller natural number, except for 1, that divides it."

Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

Also, model the same sentence in propositional logic, as detailed as possible. Clearly indicate the intended meaning of each propositional variable you use.

7.1.10 *"For all triangles it holds it is a scalene triangle iff all its sides have different lengths and all its angles have different measure."*

Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

7.1.11 *"Everyone gets a break once in a while, but the break cannot last forever"*

Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

Also, model the same sentence in propositional logic, as detailed as possible. Clearly indicate the intended meaning of each propositional variable you use.

7.1.12 Model the following sentences with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

(a) Every integer is greater or equal to one.

(b) For any two integers, their sum is smaller than their product

7.2 Syntax of Predicate Logic

7.2.1 The syntax of predicate logic is defined via 2 types of sorts: *terms* and *formulas*. What are terms and what are formulas? Give examples for both.

7.2.2 Give the definition of the syntax of predicate logic.

7.2.3 Draw the syntax tree for the following formula:

$$\forall x \left(\big(P(x,y) \to P(x,x)\big) \vee \big(Q(y,z) \wedge \exists y\, R(x,y,z)\big) \right)$$

7.2.4 Draw the syntax tree for the following formula:

$$\forall x \exists y\ (P(x, f(y)) \wedge Q(y,z) \to R(f(z))).$$

7.2.5 Given is the following formula in predicate logic

$$\varphi = \forall x \exists y \Big(\big(Q(x,y) \land P(x,y)\big) \to \big(R(y,x) \land P(x,y)\big) \Big).$$

Draw the syntax tree for φ.

7.2.6 Given is the following formula in predicate logic

$$\varphi = \exists x \forall y \Big(\big(P(x,y) \to Q(x,y)\big) \lor \big(P(y,x) \to R(x,y)\big) \Big).$$

Draw the syntax tree for φ.

7.3 Free and Bound Variables

7.3.1 Given the formula

$$P(x,y) \lor \exists y \forall x \ \big(Q(x,y) \land R(y,z)\big),$$

construct a syntax tree for φ and determine the *scope* of its quantifiers and which occurrences of the variables are *free* and which are *bound*.

7.3.2 In the context of predicate logic:

(a) What is the scope of a quantifier?
(b) What is the difference between *free* and *bound* variables?

Given an example that shows the difference.

7.3.3 In the context of predicate logic, give a definition of *substitution* of variables.

7.3.4 What does it mean to *substitute a term t for a variable x in a predicate logic formula?* Which rules to you have to consider when performing substitution? Give an example.

7.3.5 Consider the following formula.

$$\varphi := \forall y \big(P(x) \land Q(y)\big) \lor \big(R(y) \land Q(x)\big)$$

(a) Compute $\varphi[f(x)/x]$.
(b) Compute $\varphi[f(y)/x]$.
(c) Compute $\varphi[f(z)/x]$.

7.3.6 Consider the following formula.

$$\varphi := \forall y \big(P(x) \land Q(y)\big) \to \exists x \big(R(y) \land Q(x)\big)$$

(a) Compute $\varphi[f(y)/x]$.
(b) Compute $\varphi[f(x)/y]$.
(c) Compute $\varphi[k/z]$.
(d) Compute $\varphi[x/z]$.

7.3.7 Given the formula

$$\varphi = \forall x \exists z \ \big(\neg P(x) \lor Q(y, f(z))\big) \to \big(\neg \exists x \ P(y) \land Q(f(x), z)\big).$$

(a) Compute $\varphi[f(y)/x]$.
(b) Compute $\varphi[f(x)/y]$.
(c) Compute $\varphi[k/z]$.
(d) Compute $\varphi[x/z]$.

7.3.8 Given the formula

$$\varphi = \forall x \exists z \ \big(\neg P(x) \lor Q(y, f(z))\big) \to \big(\exists x \ P(y) \land Q(f(x), z)\big),$$

construct a syntax tree for φ and determine the *scope* of its quantifiers and which occurrences of the variables are *free* and which are *bound*.

7.4 Semantics of Predicate Logic

7.4.1 Give a model $\mathcal{M}$ for the following formula:

$$\varphi := \exists x \forall y P(x, y).$$

7.4.2 Consider the formula

$$\varphi := \forall x \exists y (P(x, y) \land Q(x)).$$

Give a model the satisfies the formula and a second one that falsifies the formula.

Show using the parse tree why your models satisfy are falsify the formula.

7.4.3 Consider the formula

$$\varphi = \exists x \forall y \ \big(P(x, y) \rightarrow (Q(x, y) \lor R(x, y))\big).$$

Does the following model $\mathcal{M}$ satisfy the formula?
$\mathcal{A} = \{a, b\}$
$P^{\mathcal{M}} = \{(a, a), (a, b)\}$
$Q^{\mathcal{M}} = \{(a, a), (b, a)\}$
$R^{\mathcal{M}} = \{(a, a), (b, b)\}$

7.4.4 Give the definition of a model in predicate logic. Discuss what needs to be defined in a *model* of a predicate logic formula. Give an example for each data that could be contained in a model.

7.4.5 For the following formula in *Predicate Logic*, find a *model* that satisfies the formula and one that does not. Draw a syntax tree and state all free variables while solving this task.

(a) $\forall x \exists y \ \big(P(f(y)) \ \land \ P(x)\big) \ \rightarrow \ Q(f(f(y)))$

7.4.6 In the following list, tick all items that are required for a complete model of a formula φ in predicate logic.

- ☐ A non-empty, possibly infinite set of values for variables and functions.
- ☐ A concrete value for every bound variable in φ.
- ☐ A concrete value for free bound variable in φ.
- ☐ A definition for each predicate in φ, detailing for which values/tuples the predicate returns *true*.
- ☐ A definition for each function in φ, detailing for which values/tuples the predicate returns *true*.

7.4.7 Given is the following formula in predicate logic

$$\varphi = \forall x \exists y \Big(\big(Q(x, y) \land P(x, y)\big) \rightarrow \big(R(y, x) \land P(x, y)\big) \Big)$$

and the model $\mathcal{M}$:

- $\mathcal{A} = \{a, b\}$
- $P^{\mathcal{M}} = \{(m, a) | m \in \mathcal{A}\}$
- $Q^{\mathcal{M}} = \{(b, m) | m \in \mathcal{A}\}$
- $R^{\mathcal{M}} = \{(a, b), (b, a), (b, b)\}$

Does the model $\mathcal{M}$ satisfy the formula φ? Explain your answer by drawing a **syntax tree** and evaluate the model $\mathcal{M}$ with the help of this syntax tree.

7.4.8 For the formula below, find one model that satisfies the formula, and one model that does not satisfy the formula. Explain your answer by drawing a **syntax tree** and evaluate the model $\mathcal{M}$ with the help of this syntax tree.

$$\exists x(P(x) \wedge Q(f(x))) \vee (\neg P(x) \wedge \neg Q(f(x))$$

7.4.9 For the formula below, find one model that satisfies the formula, and one model that does not satisfy the formula. Explain your answer by drawing a **syntax tree** and evaluate the model $\mathcal{M}$ with the help of this syntax tree.

$$\neg\forall x((P(x) \to P(y)) \wedge P(x))$$

7.4.10 For the formula below, state one model that satisfies the formula, and one model that does not satisfy the formula. Explain your answer by drawing a syntax tree and evaluate your models with the help of this syntax tree.

$$\forall x \exists y(P(f(x), y) \wedge \neg P(x, f(y)))$$

7.4.11 For each of the formulas in predicate logic below, find a model that satisfies the formula and one that does not. Draw a syntax tree and state all free variables while solving this task.

(a) $\neg\forall x((P(x) \to P(y)) \wedge P(x))$

(b) $\forall x \exists y(P(x, y) \wedge \neg P(f(x), f(y)))$

7.4.12 Consider the sentence $\varphi = \exists x \forall y(P(x, y) \to (Q(x, y) \vee R(x, y)))$. Does the following model satisfy φ?

The model M consists of:

- $A = \{a, b, c\}$
- $P^M = \{(a, a), (a, b), (b, a), (b, b), (c, a), (c, b)\}$
- $Q^M = \{(a, m) | m \in A\}$
- $R^M = \{(a, a), (b, a), (a, c), (b, c), (c, c)\}$

7.4.13 For each of the formulas of predicate logic below, find a model that satisfies the formula and one that does not. Draw a syntax tree and state all free variables.

(a) $\forall x(P(x, x)) \vee \forall y(Q(x, y))$

(b) $\neg\forall x((Q(f(x)) \to P(f(f(x)))) \wedge \neg Q(x))$