Questionnaire “Logic and Computability”

Summer Term 2024

Contents

4 Natural Deduction for Propositional Logic

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/
intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion. For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

4.1 Rules for Natural Deduction

4.1.1 Give the definition of a sequent. Give an example of a sequent and name the parts the sequent consists of.

4.1.2 Look at the following statements and tick them if they are true.

- ☐ In a sequent, premises entail a conclusion.
- ☐ In a sequent, conclusions entail a premise.
- ☐ A sequent is valid, independent on whether a proof can be found.
- ☐ A sequent is valid, if a proof for it can be found.

4.1.3 Explain the concept of boxes in deduction rules and why they are needed. What does it mean if you make an *assumption* within a box? Where is this assumption valid?

4.1.4 Explain the *OR-elimination* ($\vee$-*e*) rule of the natural deduction calculus. In particular, why does it rule require two boxes?

4.1.5 Why are there two rules for the $\vee$-*introduction* rule. Explain, why you are able to connect any formula to a certain formula φ using the connective $\vee$.

4.1.6 $\neg\neg\neg p \wedge q, \neg\neg r \ \vdash \ r \wedge \neg p \wedge \neg\neg q$

4.1.7 $p, p \to q, p \to (q \to r) \ \vdash \ r$

4.1.8 $\neg p \to q, \neg\neg\neg q \wedge r \ \vdash \ p \wedge \neg\neg\neg q$

4.1.9 $\neg p \to (q \to r), \neg p, \neg r \ \vdash \ \neg q$

4.1.10 $p \to q, \ q \to r \ \vdash \ p \to r$

4.1.11 $p \to (q \wedge r), q \to s \ \vdash \ p \to (s \wedge r)$

4.1.12 $p \wedge q, r \to s \ \vdash \ (p \vee (r \to s)) \wedge (q \vee ((t \vee r) \to u))$

4.1.13 $q \to r \ \vdash \ (p \vee q) \to (p \vee r)$

4.1.14 $p \to \neg q, q \ \vdash \ \neg p$

4.1.15 Solve this task without using the Modus Tollens.

$$\neg p \to \neg q, q \ \vdash \ p$$

4.1.16 $\vdash \ ((p \to q) \to p) \to p$

4.1.17 $\neg q \vee \neg p \ \vdash \ \neg(q \wedge p)$

4.1.18 $p \vee \neg\neg q, \neg p \wedge \neg q \ \vdash \ s \vee \neg t$

4.1.19 $p \to q \ \vdash \ \neg p \vee q$

4.1.20 $\vdash \ \neg(p \wedge q) \vee p$

4.1.21 $\neg\neg k \to (l \vee m), \neg\neg\neg l \to m \ \vdash \ \neg k \vee (l \vee \neg\neg m)$

4.1.22 $\neg(a \wedge b) \vee \neg c \ \vdash \ \neg(a \wedge b) \to c \vee a$

4.1.23 $(s \vee \neg u) \to t \ \vdash \ (\neg s \wedge u) \vee t$

4.1.24 Provide a natural deduction proof for the following sequent without using the *Modus Tollens* rule:

$$\varphi \to \psi, \neg\psi \ \vdash \ \neg\varphi$$

4.1.25 $\neg\neg p \wedge \neg\neg q, r \wedge s \ \vdash \ (p \wedge r) \wedge \neg\neg s$

4.1.26 $(\neg p \to q) \wedge (q \to r), \neg r \ \vdash \ \neg\neg\neg r \wedge \neg p$

4.1.27 $(p \to q) \to r \ \vdash \ \neg r \wedge \neg s \to \neg(p \to q)$

4.1.28 $p \to q \ \vdash \ (r \to p) \to (r \to q)$

4.1.29 $p \to q, p \wedge (r \vee q) \ \vdash \ (q \to p) \to ((s \wedge t) \vee q) \wedge (r \vee q)$

4.1.30 $p \vee q, \neg p \vee r \ \vdash \ q \vee r$

4.1.31 $p \to q, p \wedge r \vee q \ \vdash \ (q \to p) \to ((s \wedge t) \vee q) \wedge (r \vee q)$

4.1.32 $p \vee q, p \to r, \neg s \to \neg q \ \vdash \ r \vee s$

4.1.33 Translate the following reasoning into a sequent. If the sequent is valid, proof it using the rules of natural deduction. If the sequent is not valid, provide a counter example.

> If I press the button, the window opens.
> The window is not open.
> Therefore, I didn't press the button.

4.1.34 $\neg q \vee p \ \vdash \ q \to (p \vee r)$

4.1.35 $p \to (q \vee r), \neg q \wedge \neg r \ \vdash \ \neg p$

4.1.36 $\neg(q \vee p) \ \vdash \ \neg q \wedge p$

4.1.37 $\vdash \ (p \to q) \vee (q \to r)$

4.1.38 $(p \to q) \wedge (q \to p) \ \vdash \ (p \wedge q) \vee (\neg p \wedge \neg q)$

4.1.39 Translate the following reasoning into a sequent. If the sequent is valid, proof it using the rules of natural deduction. If the sequent is not valid, provide a counter example.

> If I press the button, the window opens.
> I pressed the button.
> Therefore, the window is open.

4.1.40 $(p \to (q \to r)) \wedge (q \to (r \to s)) \ \vdash \ p \to (q \to s)$

4.1.41 $\vdash \ (p \to (q \wedge r)) \to ((p \to q) \wedge (p \to r))$

4.1.42 $(p \to (q \to r)) \vee (q \to (r \to s)) \ \vdash \ p \to (q \to s)$

4.1.43 $\vdash (p \wedge q) \to \neg(\neg p \vee \neg q)$

4.1.44 $\neg(p \wedge q) \vee \neg(r \wedge s) \vdash (\neg p \wedge \neg r) \vee (\neg q \wedge \neg s)$

4.1.45 $p \to q \wedge \neg r, r \to s \vee \neg q \vdash p \to \neg s$

4.1.46 $p \to q \vee r, \neg q \vdash p \to r$

4.1.47 $\neg(p \wedge q) \to r, p \to \neg q \vdash r$

4.1.48 $(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$

4.1.49 $p \wedge q \to r, p \to \neg q \vdash \neg r$

4.1.50 $\vdash (p \to q) \vee \neg q$

4.1.51 $\vdash (p \to q) \wedge \neg q$

4.1.52 $(p \to q) \wedge (q \to r), \neg r \vee q \vdash \neg p \vee r$

4.1.53 $(p \wedge q) \to (p \wedge r), q \vdash \neg p \vee r$

4.1.54 $p \to q, q \to r, \neg(p \wedge r) \vdash q \to p$

4.1.55 $(p \wedge q) \to (r \vee s), (r \wedge p \wedge q) \vdash (q \to r) \wedge (s \to q)$

4.1.56 $(p \wedge r) \vdash (q \to p) \wedge (s \to r)$

4.1.57 $p \to (q \vee r), q \to s \vdash \neg(p \to (q \wedge s))$

4.1.58 $\neg(p \vee \neg q) \vdash p$

4.1.59 $p \to q \vdash ((p \vee q) \to p) \wedge (p \to (p \vee q))$

4.1.60 $p \vee q, \neg q \vee r \vdash r$

4.1.61 $(p \wedge q) \to (\neg r \wedge \neg s), \neg r \wedge \neg s \vdash p$

4.1.62 (a) If I am ill, I go to the doctor.
I am ill.
Therefore, I go to the doctor.

(b) If I am ill, I go to the doctor.
I go to the doctor.
Therefore, I am ill.

(c) (Solve without using the Modus Tollens)
If I am ill, I go to the doctor.
I did not go to the doctor.
Therefore, I am not ill.

4.1.63 (a) $(p \wedge q) \wedge \neg r \vdash q \vee r$

(b) $(p \vee q) \wedge \neg r \vdash q \wedge r$

4.1.64 $p \wedge q, q \to \neg\neg r \vdash p \wedge r$

4.1.65 $(p \wedge q) \vdash ((\neg(p \wedge q)) \to r) \vee ((q \to s) \wedge t)$

4.1.66 $\neg t, (p \wedge r) \to t \vdash (r \to s) \wedge (p \to q)$

4.1.67 $p \to q, q \to r \vdash r.$

4.1.68 Translate the following reasoning into a sequent. If the sequent is valid, proof it using the rules of natural deduction. If the sequent is not valid, provide a counter example.

If I press the button, the window opens.
The window is open.
Therefore, I pressed the button.

4.2 Soundness and Completeness of Natural Deduction

4.2.1 "Natural deduction for propositional logic is *sound* and *complete*." Explain in your own words what this means.

4.2.2 How can you show that a sequent is not valid? Is this a consequence of soundness or completeness. Explain your answer.

4.2.3 Explain what it means that natural deduction for propositional logic is *sound*. What is the difference to *completeness*?

4.2.4 Look at the following statements and tick them if they are true.

- ☐ Any sequent that is a correct semantic entailment can be proven.
- ☐ Any sequent that can be proven is a correct semantic entailment.
- ☐ If a sequent is not provable, the semantic entailment relation does hold.
- ☐ If for a sequent the semantic entailment relation does not hold, it cannot be proven with natural deduction.

4.2.5 Given an invalid sequent, how do you show its invalidity?