Questionnaire "Logic and Computability"

Summer Term 2024

Contents

8 Natural Deduction for Predicate Logic

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/
intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

8.1 Natural Deduction Rules

8.1.1 $\forall x \left(P(x) \to Q(x)\right), \forall x\ P(x) \ \vdash\ \forall x\ Q(x)$.

Solution

1.	$\forall x \left(P(x) \to Q(x)\right)$	prem.
2.	$\forall x\ P(x)$	prem.
$3x_0$	$P(x_0) \to Q(x_0)$	$\forall$e 1
4.	$P(x_0)$	$\forall$e 2
5.	$Q(x_0)$	$\to$ e 3,4
6.	$\forall x\ Q(x)$	$\forall$i 3-5

8.1.2 $\forall x\ P(x) \land \forall x\ (P(y) \to Q(x)) \quad \vdash \quad Q(z)$

Solution

1.	$\forall x\ P(x) \land \forall x\ (P(y) \to Q(x))$	prem.
2.	$\forall x\ P(x)$	$\land$e$_1$ 1
3.	$\forall x\ (P(y) \to Q(x))$	$\land$e$_2$ 1
4.	$P(y)$	$\forall$e 2
5.	$P(y) \to Q(z)$	$\forall$e 3
6.	$Q(z)$	$\to$ e 5,4

8.1.3 $\forall x(P(x) \land Q(x)) \ \vdash\ \forall x P(x) \land \forall x Q(x)$

Solution

1.		$\forall x\ (P(x) \land Q(x))$	prem.
2.	x_0	$P(x_0) \land Q(x_0)$	$\forall$e 1
3.		$P(x_0)$	$\land$e 2
4.		$\forall x\ P(x)$	$\forall$i$2-3$
5.	x_1	$P(x_1) \land Q(x_1)$	$\forall$e 1
6.		$Q(x_1)$	$\land$e 5
7.		$\forall x\ Q(x)$	$\forall$i$5-6$
8.		$\forall x\ P(x) \land \forall x\ Q(x)$	$\land$i 4,7

8.1.4 $\forall x\ P(x) \lor \forall x\ Q(x) \quad \vdash \quad \forall y\ (P(y) \lor Q(y))$

Solution

1.		$\forall x\ P(x) \lor \forall x\ Q(x)$	prem.
2.		$\forall x\ P(x)$	ass.
3.	t	$P(t)$	$\forall$e 2
4.		$P(t) \lor Q(t)$	$\lor$i$_1$ 3
5.		$\forall y\ (P(y) \lor Q(y))$	$\forall$i 3-4
6.		$\forall x\ Q(x)$	ass.
7.	s	$Q(s)$	$\forall$e 6
8.		$P(s) \lor Q(s)$	$\lor$i$_2$ 7
9.		$\forall y\ (P(y) \lor Q(y))$	$\forall$i 7-8
10.		$\forall y\ (P(y) \lor Q(y))$	$\lor$e 1,2-5,6-9

8.1.5 $\forall x\ (P(x) \to Q(y)), \forall y\ (P(y) \land R(x)) \quad \vdash \quad \exists x\ Q(x))$

Solution

1.	$\forall x\ (P(x) \to Q(y))$	prem.
2.	$\forall y\ (P(y) \land R(x))$	prem.
3.	$P(t) \to Q(y)$	$\forall$e 1
4.	$P(t) \land R(x)$	$\forall$e 2
5.	$P(t)$	$\land$e$_1$ 4
6.	$Q(y)$	$\to$ e 3
7.	$\exists x\ Q(x)$	$\exists$i 6

8.1.6 $\exists x\ \neg P(x), \forall x\ \neg Q(x) \quad \vdash \quad \exists x\ (\neg P(x) \land \neg Q(x))$

Solution

1.		$\exists x\ \neg P(x)$	prem.
2.		$\forall x\ \neg Q(x)$	prem.
3.	x_0	$\neg P(x_0)$	ass.
4.		$\neg Q(x_0)$	$\forall$e 2
5.		$\neg P(x_0) \land \neg Q(x_0)$	$\land$i 3,4
6.		$\exists x\ (\neg P(x) \land \neg Q(x))$	$\exists$i 5
7.		$\exists x\ (\neg P(x) \land \neg Q(x))$	$\exists$e 3-6

8.1.7 Consider the following natural deduction proof for the sequent

$$\forall x\ (P(x) \to Q(x)), \quad \exists x\ P(x) \quad \vdash \quad \forall x Q(x).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1.		$\forall x\ (P(x) \to Q(x))$	prem.
2.		$\exists x\ P(x)$	prem.
3.	x_0		
4.		$P(x_0)$	ass.
5.		$P(x_0) \to Q(x_0)$	$\forall$e 1
6.		$Q(x_0)$	$\to$ e, 4,5
7.		$\forall x\ Q(x)$	$\forall$i 4-6
8.		$\forall x\ Q(x)$	$\exists$e 2,3-7

Solution

This sequent is not provable.
Model $\mathcal{M}$:

$$\mathcal{A} = \{a, b\}$$
$$P^{\mathcal{M}} = \{a\}$$
$$Q^{\mathcal{M}} = \{a\}$$

$\mathcal{M} \models \forall x\ (P(x) \to Q(x)),\quad \exists x\ P(x)$
$\mathcal{M} \not\models \forall x Q(x)$

8.1.8 $\exists x\ (P(x) \to Q(y)),\quad \forall x\ P(x) \quad \vdash \quad Q(y)$

Solution

1.		$\exists x\ (P(x) \to Q(y))$	prem.
2.		$\forall x\ P(x)$	prem.
3.	x_0	$P(x_0) \to Q(y)$	ass.
4.		$P(x_0)$	$\forall$e 2
5.		$Q(y)$	$\to$ e 3,4
6.		$Q(y)$	$\exists$e 3-5

8.1.9 $\forall x\ \neg(P(x) \land Q(x)) \quad \vdash \quad \neg\exists x\ (P(x) \land Q(x))$

Solution

1.		$\forall x\ \neg(P(x) \land Q(x))$	prem.
2.		$\exists x\ (P(x) \land Q(x))$	ass.
3.	t	$P(t) \land Q(t)$	ass.
4.		$\neg P(t) \land Q(t)$	$\forall$e 1
5.		$\bot$	$\neg$e 3,4
6.		$\bot$	$\exists$e 3-5
7.		$\neg\exists x\ (P(x) \land Q(x))$	$\neg$i 2-6

8.1.10 $\exists x\ \neg P(x), \exists x\ \neg Q(x) \quad \vdash \quad \exists x\ (\neg P(x) \land \neg Q(x))$

Solution

There is no solution available for this question yet.

8.1.11 $\exists x\ (P(x) \to Q(y)), \quad \exists x\ P(x) \quad \vdash \quad Q(y)$

Solution

This sequent is not provable.
Model $\mathcal{M}$:

$$\begin{aligned}\mathcal{A} &= \{a, b\}\\ P^{\mathcal{M}} &= \{a\}\\ Q^{\mathcal{M}} &= \{a\}\\ y &\leftarrow b\end{aligned}$$

$\mathcal{M} \models \exists x\ (P(x) \to Q(y)), \quad \exists x\ P(x)$
$\mathcal{M} \not\models Q(y)$

8.1.12 $\forall x\ (P(x) \land Q(x)) \quad \vdash \quad \exists x\ (P(x) \lor Q(x))$

Solution

There is no solution available for this question yet.

8.1.13 $\forall x\ (P(x) \lor Q(x)), \quad \forall x\ (\neg P(x)) \quad \vdash \quad \forall x\ (Q(x))$

Solution

There is no solution available for this question yet.

8.1.14 $\neg\exists x\ Q(x) \quad \vdash \quad \forall x\ \neg Q(x)$

Solution

1.		$\neg\exists x\ Q(x)$	prem
2.	x_0	$Q(x_0)$	assum
3.		$\exists x\ Q(x)$	$\exists$i2
4.		$\bot$	$\neg$e1, 3
5.		$\neg Q(x_0)$	$\neg$i2 − 4
6.		$\forall x\ \neg Q(x)$	$\forall$i2 − 5

8.1.15 $\neg\exists x\ P(x) \lor \neg\exists y\ Q(y) \ \vdash\ \forall z\ \neg(Q(z) \land P(z))$

Solution

1.		$\neg\exists x\ P(x) \lor \neg\exists y\ Q(y)$	prem
2.	z_0	$Q(z_0) \land P(z_0)$	assum
3.		$\neg\exists x\ P(x)$	assum
4.		$P(z_0)$	$\land$e2
5.		$\exists x\ P(x)$	$\exists$i4
6.		$\bot$	$\neg$e3, 5
7.		$\neg\exists y\ Q(y)$	assum
8.		$Q(z_0)$	$\land$e2
9.		$\exists y\ Q(y)$	$\exists$i8
10.		$\bot$	$\neg$e7, 9
11.		$\bot$	$\lor$e1, $3-6, 7-10$
12.		$\neg(Q(z_0) \land P(z_0))$	$\neg$i$3-11$
13.		$\forall z\ \neg(Q(z) \land P(z))$	$\forall$i$3-12$

8.1.16 $\exists x\ (Q(y) \to P(x)) \quad \vdash \quad Q(y) \to \exists x\ P(x)$

Solution

1.		$\exists x\ (Q(y) \to P(x))$	prem.
2.	x_0	$Q(y) \to P(x_0)$	ass.
3.		$Q(y)$	ass.
4.		$P(x_0)$	$\to$ e 3,2
5.		$\exists x\ P(x)$	$\exists$i 4
6.		$Q(y) \to \exists x\ P(x)$	$\to$ i 3-5
7.		$Q(y) \to \exists x\ P(x)$	$\exists$e 1, 2-6

8.1.17 $\exists x\ (P(x) \to Q(x)),\ \neg Q(z)\ \vdash\ \neg P(z)$

Solution

This sequent is not provable.
Model $\mathcal{M}$:

$$\mathcal{A} = \{a, z\}$$
$$P^{\mathcal{M}} = \{a, z\}$$
$$Q^{\mathcal{M}} = \{a\}$$

$\mathcal{M} \models \exists x\ P(x) \to Q(x)$
$\mathcal{M} \models \neg Q(z)$
$\mathcal{M} \not\models \neg P(z)$

8.1.18 $\exists x\ (P(x) \wedge Q(x)) \quad \vdash \quad \exists x\ P(x) \wedge \exists x\ Q(x)$

Solution

1.		$\exists x\ (P(x) \wedge Q(x))$	prem.
2.	x_0	$P(x_0) \wedge Q(x_0)$	ass.
3.		$P(x_0)$	$\wedge$e2
4.		$Q(x_0)$	$\wedge$e2
5.		$\exists x\ P(x)$	$\exists$i3
6.		$\exists x\ Q(x)$	$\exists$i4
7.		$\exists x\ P(x) \wedge \exists\ Q(x)$	$\wedge$i$5, 6$
8.		$\exists x\ P(x) \wedge \exists x\ Q(x))$	$\exists$e$1, 2-7$

8.1.19 $\exists x\ (P(x) \vee Q(x)) \quad \vdash \quad \exists x\ P(x) \vee \exists x\ Q(x)$

Solution

There is no solution available for this question yet.

8.1.20 Explain the $\forall$-introduction rule and the $\forall$-elimination rule. Explain why one rule needs a box while the other one does not. What does it mean that x_0 needs to be fresh?

Solution

There is no solution available for this question yet.

8.1.21 $\forall x\ (P(x) \wedge Q(x)) \quad \vdash \quad \forall x\ ((Q(x) \vee R(x)) \wedge (R(x) \vee P(x)))$

Solution

1.	$\forall x\ (P(x) \wedge Q(x))$	premise
2.		fresh x_0
3.	$P(x_0) \wedge Q(x_0)$	$\forall_e$ 1 x_0
4.	$P(x_0)$	$\wedge_{e_1}$ 1
5.	$Q(x_0)$	$\wedge_{e_2}$ 1
6.	$R(x_0) \vee P(x_0)$	$\vee_{i_4}$
7.	$Q(x_0) \vee R(x_0)$	$\vee_{i_5}$
8.	$(Q(x_0) \vee R(x_0)) \wedge (R(x) \vee P(x))$	$\wedge_i$6,7
9.	$\forall x\ ((Q(x) \vee R(x)) \wedge (R(x) \vee P(x)))$	$\forall_i$ 2-8

8.1.22 $\exists x\ (Q(x) \to R(x)),\quad \exists x\ (P(x) \wedge Q(x)) \quad \vdash \quad \exists x\ (P(x) \wedge R(x))$

Solution

This sequent is not provable.
Model $\mathcal{M}$:

$$\mathcal{A} = \{a, b\}$$
$$P^{\mathcal{M}} = \{a\}$$
$$Q^{\mathcal{M}} = \{a\}$$
$$R^{\mathcal{M}} = \{\}$$

$\mathcal{M} \models \exists x(Q(x) \to R(x))$
$\mathcal{M} \models \exists x(P(x) \wedge Q(x))$
$\mathcal{M} \not\models \exists x(P(x) \wedge R(x))$

8.1.23 $\forall x\ (Q(x) \to R(x)),\quad \exists x\ (P(x) \wedge Q(x)) \quad \vdash \quad \exists x\ (P(x) \wedge R(x))$

Solution

1.	$\forall x\ (Q(x) \to R(x))$	premise
2.	$\exists x\ (P(x) \wedge Q(x))$	premise
3.	$P(x_0) \wedge Q(x_0)$	assumption fresh x_0
4.	$Q(x_0) \to R(x_0)$	$\forall_e$ 1 x_0
5.	$P(x_0)$	$\wedge_{e_1}$ 3
6.	$Q(x_0)$	$\wedge_{e_2}$ 3
7.	$R(x_0)$	$\to_e$6,4
8.	$P(x_0) \wedge R(x_0)$	$\wedge_i$5,7
9.	$\exists x\ (P(x) \wedge R(x))$	$\exists_i$ 8
10.	$\exists x\ (P(x) \wedge R(x))$	$\exists_e$ 2,3-9

8.1.24 $\neg\exists x \forall y\ (P(x) \wedge Q(y)) \quad \vdash \quad \forall x \exists y\ \neg(P(x) \wedge Q(y))$

Solution

1.	$\neg\exists x\forall y\ (P(x) \wedge Q(y))$	premise
2.	$\forall y(P(x_0) \wedge Q(y))$	assumption fresh x_0
3.	$\exists x\forall y(P(x) \wedge Q(y))$	$\exists_i 2$
4.	$\bot$	$\neg_e 1, 3$
5.	$\neg\forall y(P(x_0) \wedge Q(y))$	$\neg_i 2-4$
6.	$\neg\exists y\neg(P(x_0) \wedge Q(y))$	assumption
7.	$\neg(P(x_0) \wedge Q(y_0))$	assumption fresh y_0
8.	$\exists y\neg(P(x_0) \wedge Q(y))$	$\exists_i 7$
9.	$\bot$	$\neg_e 6, 8$
10.	$P(x_0) \wedge Q(y_0)$	$PBC 7-9$
11.	$\forall y(P(x_0) \wedge Q(y))$	$\forall_i 7-10$
12.	$\bot$	$\neg_e 5, 11$
13.	$\exists y\neg(P(x_0) \wedge Q(y))$	$PBC 6-12$
14.	$\forall x\exists y\ \neg(P(x) \wedge Q(y))$	$\forall_i 2-13$

8.1.25 $\forall x\exists y\ \neg(P(x) \wedge Q(y)) \quad \vdash \quad \neg\exists x\forall y\ (P(x) \wedge Q(y))$

Solution

1.	$\forall x\exists y\ \neg(P(x) \wedge Q(y))$	premise
2.	$\exists y\neg(P(x_0) \wedge Q(y))$	$\forall_e 1$
3.	$\neg(P(x_0) \wedge Q(y_0))$	assumption fresh y_0
4.	$\forall y(P(x_0) \wedge Q(y))$	assumption
5.	$P(x_0) \wedge Q(y_0)$	$\forall_e 4$
6.	$\bot$	$\neg_e 3, 5$
7.	$\neg\forall y(P(x_0) \wedge Q(y))$	$\neg_i 4-6$
8.	$\neg\forall y(P(x_0) \wedge Q(y))$	$\exists_e 2, 3-7$
9.	$\exists x\forall y(P(x) \wedge Q(y))$	assumption
10.	$\forall y(P(x_0) \wedge Q(y))$	assumption fresh x_0
11.	$\bot$	$\neg_e 8, 10$
12.	$\bot$	$\exists_e 9, 10-11$
13.	$\neg\exists x\forall y\ (P(x) \wedge Q(y))$	$\neg_i 9-12$

8.1.26 $\neg\exists x\ \neg P(x) \quad \vdash \quad \forall x\ \neg P(x)$

Solution

This sequent is not provable.
Model $\mathcal{M}$:

$$\mathcal{A} = \{a\}$$
$$P^{\mathcal{M}} = \{a\}$$

$\mathcal{M} \models \neg\exists x\neg P(x)$
$\mathcal{M} \not\models \forall x\neg P(x)$

8.1.27 $P(x) \vee Q(y), \quad P(x) \to R(z), \quad Q(y) \to R(z) \quad \vdash \quad R(z)$

Solution

1.	$P(x) \vee Q(y)$	premise
2.	$P(x) \to R(z)$	premise
3.	$Q(y) \to R(z)$	premise
4.	$P(x)$	assumption
5.	$R(z)$	$\to_e 2, 4$
6.	$Q(y)$	assumption
7.	$R(z)$	$\to_e 3, 6$
8.	$R(z)$	$\vee_e 1, 4-5, 6-7$

8.1.28 $\exists y \forall x\ (P(x,y)) \quad \vdash \quad \forall x \exists y\ (P(x,y))$

Solution

This sequent is provable.

1.	$\exists y \forall x P(x,y)$	premise
2.	$\forall x P(x, y_0)$	assumption fresh y_0
3.	$P(x_0, y_0)$	$\forall_e 2$ fresh x_0
4.	$\exists y P(x_0, y)$	$\exists_i 3$
5.	$\forall x \exists y P(x,y)$	$\forall_i 3-4$
6.	$\forall x \exists y P(x,y)$	$\exists_e 1, 2-5$

8.1.29 $\exists a \forall b\ (S(b,a) \wedge T(b,a)) \quad \vdash \quad \forall b \forall a\ (S(b,a) \wedge T(b,a))$

Solution

This sequent is not provable.
Model $\mathcal{M}$:

$$\mathcal{A} = \{0, 1\}$$
$$S^{\mathcal{M}} = \{(0,1), (1,1)\}$$
$$T^{\mathcal{M}} = \{(0,1), (1,1)\}$$

$\mathcal{M} \models \exists a \forall b\ (S(b,a) \wedge T(b,a))$
$\mathcal{M} \not\models \forall b \forall a\ (S(b,a) \wedge T(b,a))$

8.1.30 $P(y) \to \forall x Q(x), \ \exists x \neg Q(x) \ \vdash \ \exists x \neg P(x)$

Solution

1.	$P(y) \to \forall x Q(x)$	premise
2.	$\exists x \neg Q(x)$	premise
3.	$P(y)$	assumption
4.	$\forall x Q(x)$	$\to_e$ 1, 3
5.	$\neg Q(x_0)$	assumption fresh x_0
6.	$Q(x_0)$	$\forall_e 4$
7.	$\bot$	$\neg_e 5, 6$
8.	$\bot$	$\exists_e 2, 5-7$
9.	$\neg P(y)$	$\neg_i 3-8$
10.	$\exists x \neg P(x)$	$\exists_i 9$

8.1.31 Consider the following natural deduction proof for the sequent

$$\exists x\ \neg P(x) \quad \vdash \quad \neg\forall x\ P(x).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1.	$\exists x\ \neg P(x)$	prem.
2.	$\forall x\ P(x)$	ass.
3.	$P(x_0)$	∀e 2
4.	$\exists x\ P(x)$	∃i 3
5.	$\bot$	¬e 1,4
6.	$\neg\forall x\ P(x)$	¬e 2-5

Solution

There is no solution available for this question yet.

8.1.32 Consider the following natural deduction proof for the sequent

$$\exists x\ P(x) \vee \exists x\ Q(x) \quad \vdash \quad \exists x\ (P(x) \vee Q(x)).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1.		$\exists x\ P(x) \vee \exists x\ Q(x)$	prem.
2.		$\exists x\ P(x)$	ass.
3.	x_0	$P(x_0)$	ass.
4.		$P(x_0) \vee Q(x_0)$	$\vee i_1$ 3
5.		$\exists x\ (P(x) \vee Q(x))$	∃e 2,3-4
6.		$\exists x\ Q(x)$	ass.
7.	x_0	$Q(x_0)$	ass.
8.		$P(x_0) \vee Q(x_0)$	$\vee i_2$ 7
9.		$\exists x\ (P(x) \vee Q(x))$	∃e 6,7-8
10.		$\exists x\ (P(x) \vee Q(x))$	∨e 1,2-5,6-9

Solution

1.	$\exists x\ P(x) \vee \exists x\ Q(x)$	premise
2.	$\exists x\ P(x)$	assumption
3.	$P(x_0)$	assumption fresh x_0
4.	$P(x_0) \vee Q(x_0)$	$\vee_{i_3}$
5.	$\exists x\ (P(x) \vee Q(x))$	$\exists_i 4$
6.	$\exists x\ (P(x) \vee Q(x))$	$\exists_e 2, 3-5$
7.	$\exists x\ Q(x)$	assumption
8.	$Q(x_0)$	assumption fresh x_0
9.	$P(x_0) \vee Q(x_0)$	$\vee_{i_8}$
10.	$\exists x\ (P(x) \vee Q(x))$	$\exists_i 9$
11.	$\exists x\ (P(x) \vee Q(x))$	$\exists_e 7, 8-10$
12.	$\exists x\ (P(x) \vee Q(x))$	$\vee_e 1, 2-6, 7-11$

8.1.33 $\forall x \exists y\ (P(x) \rightarrow Q(y)), P(s) \quad \vdash \quad \exists x \forall y\ (\neg P(x) \vee Q(y))$

Solution

This sequent is not provable.
Model $\mathcal{M}$:

$$\mathcal{A} = \{a, b\}$$
$$P^{\mathcal{M}} = \{a, b\}$$
$$Q^{\mathcal{M}} = \{a\}$$

$\mathcal{M} \models \forall x \exists y\ (P(x) \rightarrow Q(y)), P(s)$
$\mathcal{M} \not\models \exists x \forall y\ (\neg P(x) \vee Q(y))$

8.1.34 $\forall a \forall b\ (P(a) \wedge Q(b)) \quad \vdash \quad \forall a \exists b\ (P(a) \vee Q(b))$

Solution

There is no solution available for this question yet.

8.1.35 $\exists x\ \neg P(x) \quad \vdash \quad \neg \forall x\ P(x).$

Solution

There is no solution available for this question yet.