

Questionnaire “Logic and Computability”

Summer Term 2024

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8 Natural Deduction for Predicate Logic

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

8.1 Natural Deduction Rules

$$8.1.1 \quad \forall x (P(x) \rightarrow Q(x)), \forall x P(x) \vdash \forall x Q(x).$$

$$8.1.2 \quad \forall x P(x) \wedge \forall x (P(y) \rightarrow Q(x)) \vdash Q(z)$$

$$8.1.3 \quad \forall x (P(x) \wedge Q(x)) \vdash \forall x P(x) \wedge \forall x Q(x)$$

$$8.1.4 \quad \forall x P(x) \vee \forall x Q(x) \vdash \forall y (P(y) \vee Q(y))$$

$$8.1.5 \quad \forall x (P(x) \rightarrow Q(y)), \forall y (P(y) \wedge R(x)) \vdash \exists x Q(x)$$

$$8.1.6 \quad \exists x \neg P(x), \forall x \neg Q(x) \vdash \exists x (\neg P(x) \wedge \neg Q(x))$$

8.1.7 Consider the following natural deduction proof for the sequent

$$\forall x (P(x) \rightarrow Q(x)), \exists x P(x) \vdash \forall x Q(x).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

| | | |
|----|-------------------------------------|-----------------------|
| 1. | $\forall x (P(x) \rightarrow Q(x))$ | prem. |
| 2. | $\exists x P(x)$ | prem. |
| 3. | x_0 | |
| 4. | $P(x_0)$ | ass. |
| 5. | $P(x_0) \rightarrow Q(x_0)$ | $\forall e$ 1 |
| 6. | $Q(x_0)$ | $\rightarrow e$, 4,5 |
| 7. | $\forall x Q(x)$ | $\forall i$ 4-6 |
| 8. | $\forall x Q(x)$ | $\exists e$ 2,3-7 |

$$8.1.8 \quad \exists x (P(x) \rightarrow Q(y)), \forall x P(x) \vdash Q(y)$$

$$8.1.9 \quad \forall x \neg (P(x) \wedge Q(x)) \vdash \neg \exists x (P(x) \wedge Q(x))$$

$$8.1.10 \quad \exists x \neg P(x), \exists x \neg Q(x) \vdash \exists x (\neg P(x) \wedge \neg Q(x))$$

$$8.1.11 \quad \exists x (P(x) \rightarrow Q(y)), \exists x P(x) \vdash Q(y)$$

$$8.1.12 \quad \forall x (P(x) \wedge Q(x)) \vdash \exists x (P(x) \vee Q(x))$$

$$8.1.13 \quad \forall x (P(x) \vee Q(x)), \forall x (\neg P(x)) \vdash \forall x (Q(x))$$

$$8.1.14 \quad \neg \exists x Q(x) \vdash \forall x \neg Q(x)$$

$$8.1.15 \quad \neg \exists x P(x) \vee \neg \exists y Q(y) \vdash \forall z \neg (Q(z) \wedge P(z))$$

$$8.1.16 \quad \exists x (Q(y) \rightarrow P(x)) \vdash Q(y) \rightarrow \exists x P(x)$$

$$8.1.17 \quad \exists x (P(x) \rightarrow Q(x)), \neg Q(z) \vdash \neg P(z)$$

8.1.18 $\exists x (P(x) \wedge Q(x)) \quad \vdash \quad \exists x P(x) \wedge \exists x Q(x)$

8.1.19 $\exists x (P(x) \vee Q(x)) \quad \vdash \quad \exists x P(x) \vee \exists x Q(x)$

8.1.20 Explain the \forall -introduction rule and the \forall -elimination rule. Explain why one rule needs a box while the other one does not. What does it mean that x_0 needs to be fresh?

8.1.21 $\forall x (P(x) \wedge Q(x)) \quad \vdash \quad \forall x ((Q(x) \vee R(x)) \wedge (R(x) \vee P(x)))$

8.1.22 $\exists x (Q(x) \rightarrow R(x)), \quad \exists x (P(x) \wedge Q(x)) \quad \vdash \quad \exists x (P(x) \wedge R(x))$

8.1.23 $\forall x (Q(x) \rightarrow R(x)), \quad \exists x (P(x) \wedge Q(x)) \quad \vdash \quad \exists x (P(x) \wedge R(x))$

8.1.24 $\neg \exists x \forall y (P(x) \wedge Q(y)) \quad \vdash \quad \forall x \exists y \neg (P(x) \wedge Q(y))$

8.1.25 $\forall x \exists y \neg (P(x) \wedge Q(y)) \quad \vdash \quad \neg \exists x \forall y (P(x) \wedge Q(y))$

8.1.26 $\neg \exists x \neg P(x) \quad \vdash \quad \forall x \neg P(x)$

8.1.27 $P(x) \vee Q(y), \quad P(x) \rightarrow R(z), \quad Q(y) \rightarrow R(z) \quad \vdash \quad R(z)$

8.1.28 $\exists y \forall x (P(x, y)) \quad \vdash \quad \forall x \exists y (P(x, y))$

8.1.29 $\exists a \forall b (S(b, a) \wedge T(b, a)) \quad \vdash \quad \forall b \forall a (S(b, a) \wedge T(b, a))$

8.1.30 $P(y) \rightarrow \forall x Q(x), \quad \exists x \neg Q(x) \quad \vdash \quad \exists x \neg P(x)$

8.1.31 Consider the following natural deduction proof for the sequent

$$\exists x \neg P(x) \quad \vdash \quad \neg \forall x P(x).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

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|----|-----------------------|---------------|
| 1. | $\exists x \neg P(x)$ | prem. |
| 2. | $\forall x P(x)$ | ass. |
| 3. | $P(x_0)$ | $\forall e$ 2 |
| 4. | $\exists x P(x)$ | $\exists i$ 3 |
| 5. | \perp | $\neg e$ 1,4 |
| 6. | $\neg \forall x P(x)$ | $\neg e$ 2-5 |

8.1.32 Consider the following natural deduction proof for the sequent

$$\exists x P(x) \vee \exists x Q(x) \quad \vdash \quad \exists x (P(x) \vee Q(x)).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

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|-----|--------------------------------------|--------------------|
| 1. | $\exists x P(x) \vee \exists x Q(x)$ | prem. |
| 2. | $\exists x P(x)$ | ass. |
| 3. | $x_0 \quad P(x_0)$ | ass. |
| 4. | $P(x_0) \vee Q(x_0)$ | $\vee i_1$ 3 |
| 5. | $\exists x (P(x) \vee Q(x))$ | $\exists e$ 2,3-4 |
| 6. | $\exists x Q(x)$ | ass. |
| 7. | $x_0 \quad Q(x_0)$ | ass. |
| 8. | $P(x_0) \vee Q(x_0)$ | $\vee i_2$ 7 |
| 9. | $\exists x (P(x) \vee Q(x))$ | $\exists e$ 6,7-8 |
| 10. | $\exists x (P(x) \vee Q(x))$ | $\vee e$ 1,2-5,6-9 |

$$8.1.33 \quad \forall x \exists y (P(x) \rightarrow Q(y)), P(s) \quad \vdash \quad \exists x \forall y (\neg P(x) \vee Q(y))$$

$$8.1.34 \quad \forall a \forall b (P(a) \wedge Q(b)) \quad \vdash \quad \forall a \exists b (P(a) \vee Q(b))$$

$$8.1.35 \quad \exists x \neg P(x) \quad \vdash \quad \neg \forall x P(x).$$