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A Method to Multi-Attribute Group Decision-Making Problem with Complex q-Rung Orthopair Linguistic Information Based on Heronian Mean Operators

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ABSTRACT

The notions of complex q-rung orthopair fuzzy sets (Cq-ROFSs) and linguistic sets (LSs) are two different concepts to deal with uncertain information in multi-attribute group decision-making (MAGDM) problems. The Heronain mean (HM) and geometric Heronain mean (GHM) operators are an effective tool used to aggregate some q-rung orthopair linguistic fuzzy numbers (q-ROLFNs) into a single element. The purpose of this manuscript is to propose a new concept called complex q-rung orthopair linguistic sets (Cq-ROLSs) to cope with complex uncertain information in real decision-making problems. Then the fundamental laws and their examples of the Cq-ROLSs are also given. Furthermore, the notions of complex q-rung orthopair linguistic Heronian mean (Cq-ROLHM) operator, complex q-rung orthopair linguistic weighted Heronian mean (Cq-ROLWHM) operator, complex q-rung orthopair linguistic geometric Heronian mean (Cq-ROLGHM) operator, complex q-rung orthopair linguistic weighted geometric Heronian mean (Cq-ROLWGHM) operator are proposed and their basic properties are also discussed. Moreover, we develop a novel approach to MAGDM using proposed operators and a numerical example is used to describe the flexibility and explicitly of the initiated operators. In last, the comparison between proposed method and existing work is also discussed in detail.

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1. INTRODUCTION

The framework of the complex fuzzy set (CFS) was proposed by Ramot *et al.* [1], which is a generaeronian mean lization of a fuzzy set (FS) [2]. The difference between CFS and FS is that the range of CFS is not restricted to [0, 1], but is extended into a unit disc in a complex plane. The CFS has received more attention in the environment of FS theory. While, Alkouri and Salleh [3] proposed the notions of linguistic variable, hedges and several distances on CFS. Yazdanbakhsh and Dick [4] proposed time-series forecasting via complex fuzzy logic and a systematic review of CFS. Recently, Bi *et al.* [5] proposed complex fuzzy geometric aggregation operators. Because of its merits and advantages, CFS has been extensively applied to decision-making problems and other fields [6–7]. Because FS and CFS can only describe the membership degree and complex-valued membership degree, and cannot express the non-membership degree and complex-valued non-membership degree. Then the framework of intuitionistic fuzzy set (IFS) is introduced by Atanassov [8] as a generalization of FS by including non-membership degree. The IFS is characterized by two different degrees such as membership and non-membership grades, and their sum is limited to [0, 1]. IFS has been extensively used in different fields [9–10]. Further, Alkouri *et al.* [11] proposed the framework of complex intuitionistic fuzzy set (CIFS) as a generalization of FS to deal with uncertain and unpredictable information in real-life problems. The CIFS is characterized by complex-valued membership grade and complex-valued non-membership grade in the form of polar coordinates. Because there is a restrict condition in IFS, further, Yager [12] initiated the idea of Pythagorean fuzzy set (PFS) as an effective tool to describe the uncertainty for the multi-attribute group decision making (MAGDM) problems. The notion of PFS is more general than IFS and FS to cope with difficult information in real decision problems. When a decision maker provides (0.6,0.7) for membership and non-membership grades, i.e., $0.6 + 0.7 = 1.3 \geq 1$, the IFS cannot describe it effectively, but the PFS can describe such kinds of information effectively, i.e., $0.6^2 + 0.7^2 = 0.36 + 0.49 = 0.85 < 1$. Based on PFS, Garg [13,14] proposed a novel correlation coefficient between PFSs, and a new generalized Pythagorean fuzzy Einstein aggregation operators and their application to decision-making were developed. Dick *et al.* [15] introduced Pythagorean and complex fuzzy operations. Garg [16] further proposed a novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multi-attribute decision-making (MADM) problem. Further, some new MADM methods and

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operators about PFSs were developed. Ren *et al.* [17] developed the notion of Pythagorean fuzzy TODIM (an acronym in Portuguese for Interactive Multi-Criteria Decision-Making) approach to MADM. Garg [18] proposed a new improved score function of an interval-valued PFSs based TOPSIS method. Wei and Wei [19] proposed the concept of similarity measures for PFSs based on the cosine function and their applications. Garg [20], proposed generalized Pythagorean fuzzy geometric aggregation operators using Einstein t-norm and t-conorm for the multi-criteria decision-making process. Wei [21] proposed the Pythagorean fuzzy interaction aggregation operators and their application to MADM problems.

In some particular cases, the IFS and PFS are failed, if a decision maker provides $(0.9, 0.7)$ for membership and non-membership degrees, i.e., $0.9+0.7 = 1.6 \geq 1$ and $0.9^2+0.7^2 = 0.81+0.49 = 1.30 \geq 1$, the IFS and PFS cannot describe effectively such kinds of information. To precisely cope with such kind of problems, Yager [22] proposed the framework of q-rung orthopair fuzzy set (q-ROFS) whose restriction is that the sum of q-power of membership and q-power of non-membership grade is belonging to [0,1]. Obviously, the q-ROFS can describe effectively such kinds of information, i.e., $0.9^4+0.7^4 = 0.7+0.24 = 0.94 \leq 1$. The FS, IFS, and PFS all are the special cases of q-ROFS, this characteristic makes q-ROFS more general than existing FSs. For example, if q=1 and non-membership equals to zero then, the q-ROFS is converted to FS. If q=1, then the q-ROFS is converted to IFS. If q=2, then the q-ROFS is converted to PFS. To better understand the relationship among q-ROFS, PFS, and IFS, please see Figure 1.

Further, Liu and Liu [23] initiated the concept of some q-rung orthopair fuzzy Bonferroni mean operators and their application to MAGDM. Wei *et al.* [24] proposed some q-rung orthopair fuzzy Heronian mean (HM) operators. Liu and Wang [25,26] proposed some q-rung orthopair fuzzy aggregation operators and their application to MADM based on Archimedean Bonferroni operators of q-rung orthopair fuzzy numbers. Further, the power maclaurin symmetric mean [27], partitioned maclaurin symmetric mean [28], as a powerful operator to aggregate the interrelation among q-ROFNs, were developed. Li *et al.* [29] proposed q-rung orthopair linguistic HM operators with their application to MAGDM.

Moreover, the linguistic variable (LV), proposed by Zadeh [30], can easily express the qualitative information. Many researchers combined the notion of LV with IFS, PFS, q-ROFS, and proposed the novel concepts of intuitionistic linguistic fuzzy numbers [31], linguistic Pythagorean FS [32] and q-rung orthopair linguistic HM operators [29]. Obviously, these combinations can easily describe the complex fuzzy information.

Consequently, motivated by the idea from IFS to CIFS, it is necessary to extend q-ROFS to complex q-ROFS (Cq-ROFS) because Cq-ROFS is a powerful idea to cope with uncertain and unpredictable information, and it is also a generalization of CFS and FS, whose constraint is like q-ROFS, but the range of membership and non-membership grades are bounded to unit disc in a complex plane instead of [0,1]. The complex-valued membership and complex-valued non-membership grades are represented in the polar form. The q-ROFS copes with one-dimension information at a time in a single elements, which results in data loss sometimes. But, the Cq-ROFS is a powerful tool to deal with uncertain information as compared to q-ROFS, because it contains two-dimension information in a single elements. So by introducing the second dimension to the grade of membership and non-membership, loss of data can be avoided. At the same time, motivated by combining the LV with q-ROFS, it is meaningful to combine the LV with Cq-ROFS, and propose complex q-rung orthopair linguistic number (Cq-ROLN) which is more general than existing fuzzy sets, such as complex Pythagorean linguistic set (CPYLS) and complex intuitionistic linguistic set (CILS). If we take the imaginary part is zero, in the terms of membership grade and non-membership grade, then the Cq-ROLS is convert into q-rung orthopair linguistic set (q-ROLS), i.e., q-ROLS is its special case. If we set the value of parameter $q = 1$ in the environment of q-ROLS, then the q-ROLS is converted for intuitionistic linguistic set (ILS). Similarly, if we take the value of parameter $q = 2$ in the environment of q-ROLS, then the q-ROLS is converted for Pythagorean linguistic set (PYLS). The ILS and PYLS are the particular

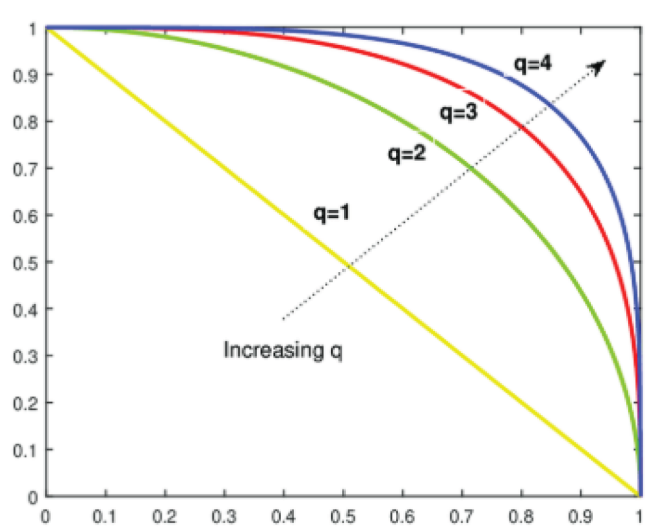

Figure 1 | Geometrical interpretation of q-rung orthopair fuzzy set.

cases of the Cq-ROLSs. Moreover, Heronain mean (HM) can consider the relationship between any two attributes, compared with the BM which has the same function as HM, however, HM can reduce the operational amount to half of BM. So it is necessary to extend HM to Cq-ROFS and Cq-ROLN, and then to propose a new MADM method based on the proposed operators. Therefore, the motivation and goal of this paper are shown as follows:

1. Propose the notion of Cq-ROFS and some operational laws, and then explain their characteristics and comparison method.
2. Propose the notion of Cq-ROLN and some operational laws, and then explain their characteristics and comparison method.
3. Develop some extended HM operators, such as complex q-rung orthopair linguistic Heronian mean operator (Cq-ROLHM), complex q-rung orthopair linguistic weighted Heronian mean operator Cq-(ROLWHM), complex q-rung orthopair linguistic geometric Heronian mean operator (Cq-ROLGHM), complex q-rung orthopair linguistic weighted geometric Heronian mean operator(Cq-ROLWGHM), and then verify their some properties.
4. Develop a new MADM method based on the proposed operators.
5. Give some examples to show the flexibility and superiority of the developed method.

The construct of this manuscript is as follows: in Section 1, we introduce some basic theories, and propose notion of Cq-ROFS; Section 3 proposes the notion of Cq-ROLN and their operational laws. In Section 4, we propose the Cq-ROLHM, Cq-ROLWHM, Cq-ROLGHM, Cq-ROLWGHM operators and their properties. In Section 5, we propose a new method to solve MAGDM problem based on the proposed operators. In Section 6, some examples are given to show the flexibility and superiority of our proposed operators. The conclusion is discussed in Section 7.

2. PRELIMINARIES

In this section, we will review the existing concepts and initiate the idea of Cq-ROFSs. The operational laws of Cq-ROFSs are also discussed in detail.

2.1. The q-ROFS

In this sub-section, we review some basic concepts of q-ROFS, LV, HM, GHM and their operations.

Definition 1. [22] For ordinary fixed set X, the q-ROFS is given by

$$P = \left\{ \left\langle x, \mathfrak{t}'(x), \mathfrak{f}'(x) \right\rangle / x \in X \right\} \tag{1}$$

where $\mathfrak{t}'(x), \mathfrak{f}'(x) : X \rightarrow [0,1]$ denoted the membership and non-membership degrees respectively, satisfying the condition $0 \leq \mathfrak{t}'^q(x) + \mathfrak{f}'^q(x) \leq 1\,(q \geq 1)$. The hesitancy degree is defined by $\mu_P(x) = \left(\mathfrak{t}'^q(x) + \mathfrak{f}'^q(x) - \mathfrak{t}'^q(x)\,\mathfrak{f}'^q(x)\right)^{\frac{1}{q}}$. Further, $\left(\mathfrak{t}'(x), \mathfrak{f}'(x)\right)$ is called q-ROFN. The geometrical interpretation of Cq-ROFS is shown in Figure 1.

Definition 2. [22] For two q-ROFNs $P_1 = \left(\mathfrak{t}'_1(x), \mathfrak{f}'_1(x)\right)$ and $P_2 = \left(\mathfrak{t}'_2(x), \mathfrak{f}'_2(x)\right)$, their operational laws are defined by ($\delta \geq 1$ is a positive number)

1.

$$P_1 \otimes P_2 = \left(\left(\mathfrak{t}'^q_1 + \mathfrak{t}'^q_2 - \mathfrak{t}'^q_1 \mathfrak{t}'^q_2\right)^{\frac{1}{q}}, \mathfrak{f}'_1 \mathfrak{f}'_2 \right) \tag{2}$$

2.

$$P_1 \otimes P_2 = \left(\mathfrak{t}'_1 \mathfrak{t}'_2, \left(\mathfrak{f}'^q_1 + \mathfrak{f}'^q_2 - \mathfrak{f}'^q_1 \mathfrak{f}'^q_2\right)^{\frac{1}{q}} \right) \tag{3}$$

3.

$$\delta P_1 = \left(\left(1 - \left(1 - \mathfrak{t}'^q_1\right)^{\delta}\right)^{\frac{1}{q}}, \mathfrak{f}'^{\delta}_1 \right) \tag{4}$$

4.

$$P_1^{\mathcal{S}} = \left(\mathfrak{t}_1'^{\mathcal{S}}, (1- \left(1 - \mathfrak{f}_1'^{q}\right)^{\mathcal{S}}\right)^{\frac{1}{q}} \tag{5}$$

Definition 3. [22] For q-ROFS $P = \left(\mathfrak{t}'(x), \mathfrak{f}'(x)\right)$, the score and accuracy functions are defined by

$$S(P) = \left(\mathfrak{t}'^{q} - \mathfrak{f}'^{q}\right) \tag{6}$$

$$H(P) = \left(\mathfrak{t}'^{q} + \mathfrak{f}'^{q}\right) \tag{7}$$

For any two q-ROFNs $P_1 = \left(\mathfrak{t}_1'(x), \mathfrak{f}_1'(x)\right)$ and $P_2 = \left(\mathfrak{t}_2'(x), \mathfrak{f}_2'(x)\right)$, then we have

1. If $S(P_1) > S(P_2)$, then $P_1 > P_2$.
2. If $S(P_1) = S(P_2)$, then
    1. If $H(P_1) > H(P_2)$, then If $P_1 > P_2$.
    2. If $H(P_1) = H(P_2)$, then If $P_1 = P_2$.

2.2. Linguistic Term Set and HM

Definition 4. [30] For a linguistic term set $\mathcal{S} = \{\mathcal{S}_i / i = 1, 2, .., z\}$ with odd cardinality, where, z is the cardinality of $\mathcal{S}$, and $\mathcal{S}_i$ is a linguistic variable. A possible linguistic term set is given by

$\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6, \mathcal{S}_7\} = \{$*very poor, poor, slightly poor, fair, slightly good, good, very good*$\}$. The linguistic terms are expressed by PFSs for 5 or 7 terms, which are shown in Figure 2.

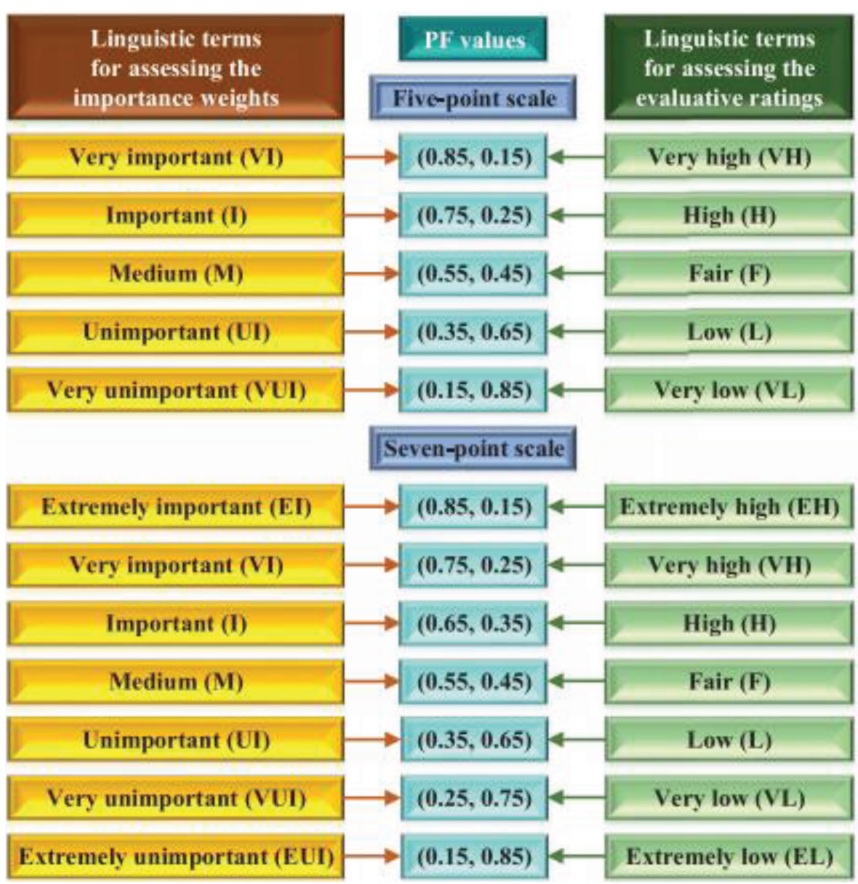

Figure 2 A linguistic rating system for constructing Pythagorean fuzzy data.

HM and geometric Heronian mean (GHM) are a more generalized operators than existing operators like averaging mean operator, geometric mean operator, weighted averaging mean operator, weighted geometric mean operator and more others. The operators which are discussed in [5,20,24,25,29,31,33,34] are all the special cases of the proposed operators. In this article, we will use HM and GHM operators to propose the complex q-rung orthopair linguistic Heronian mean and complex q-rung orthopair linguistic geometric Heronian mean operators.

Definition 5. [29] For a set of crisp numbers $P_i\,(i = 1, 2, .., n)$ with$s, t > 0$, Heronian mean (HM) is given by

$$HM^{s,t}\left(P_1, P_2, .., P_n\right) = \left(\frac{2}{n\,(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} P_i^s P_j^t\right)^{\frac{1}{s+t}} \tag{8}$$

Definition 6. [29] For a family of crisp numbers $P_i\,(i = 1, 2, .., n)$ with$s, t > 0$, the Geometric Heronian mean (GHM) is given by

$$GHM^{s,t}\left(P_1, P_2, .., P_n\right) = \left(\frac{1}{s+t} \prod_{i=1}^{n} \prod_{j=i}^{n} \left(sP_i + tP_j\right)^{\frac{2}{n(n+1)}}\right) \tag{9}$$

2.3. The Complex q-Rung Orthopair Fuzzy Set

In this section, we will propose the notion of complex q-rung orthopair fuzzy set (Cq-ROFS) and their operations.

Definition 7. For ordinary fixed set X, the Cq-ROFS is given by

$$P = \left\{\left\langle x, \mathfrak{t}'\,(x), \mathfrak{f}'\,(x)\right\rangle / x \in X\right\} \tag{10}$$

where $\mathfrak{t}'\,(x) = \mathfrak{t}\,(x)\, e^{i2\pi \mathcal{W}_{\mathfrak{t}(x)}}$ and $\mathfrak{f}'\,(x) = \mathfrak{f}\,(x)\, e^{i2\pi \mathcal{W}_{\mathfrak{f}(x)}}$ denoted complex-valued membership and non-membership degrees respectively, satisfying the condition $0 \le \mathfrak{t}^q\,(x) + \mathfrak{f}^q\,(x) \le 1$ and $0 \le \mathcal{W}_{\mathfrak{t}(x)}^q + \mathcal{W}_{\mathfrak{f}(x)}^q \le 1, (q \ge 1)$. The hesitancy degree is defined by $\mu\,(x) = \left(1 - \left(\mathfrak{t}^q\,(x) + \mathfrak{f}^q\,(x)\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(\mathcal{W}_{\mathfrak{t}(x)}^q + \mathcal{W}_{\mathfrak{f}(x)}^q\right)\right)^{\frac{1}{q}}}$. Moreover, $\left(\mathfrak{t}\,(x)\, e^{i2\pi \mathcal{W}_{\mathfrak{t}(x)}}, \mathfrak{f}\,(x)\, e^{i2\pi \mathcal{W}_{\mathfrak{f}(x)}}\right)$ is called complex q-rung orthopair fuzzy number (Cq-ROFN). Simply we write $\left(\mathfrak{t} e^{i2\pi \mathcal{W}_{\mathfrak{t}}}, \mathfrak{f} e^{i2\pi \mathcal{W}_{\mathfrak{f}}}\right)$.

Definition 8. For two Cq-ROFNs $P_1 = \left(\mathfrak{t}_1 e^{i2\pi \mathcal{W}_{\mathfrak{t}_1}}, \mathfrak{f}_1 e^{i2\pi \mathcal{W}_{\mathfrak{f}_1}}\right)$ and $P_2 = \left(\mathfrak{t}_2 e^{i2\pi \mathcal{W}_{\mathfrak{t}_2}}, \mathfrak{f}_2 e^{i2\pi \mathcal{W}_{\mathfrak{f}_2}}\right)$, the operational laws are defined by ($\delta \ge 1$ is a positive number)

1.

$$P_1 \otimes P_2 = \left(\left(\mathfrak{t}_1^q + \mathfrak{t}_2^q - \mathfrak{t}_1^q \mathfrak{t}_2^q\right)^{\frac{1}{q}} e^{i2\pi\left(\mathcal{W}_{\mathfrak{t}_1}^q + \mathcal{W}_{\mathfrak{t}_2}^q - \mathcal{W}_{\mathfrak{t}_1}^q \mathcal{W}_{\mathfrak{t}_2}^q\right)^{\frac{1}{q}}}, \mathfrak{f}_1 \mathfrak{f}_2 e^{i2\pi \mathcal{W}_{\mathfrak{f}_1} \mathcal{W}_{\mathfrak{f}_2}}\right) \tag{11}$$

2.

$$P_1 \otimes P_2 = \left(\mathfrak{t}_1 \mathfrak{t}_2 e^{i2\pi \mathcal{W}_{\mathfrak{t}_1} \mathcal{W}_{\mathfrak{t}_2}}, \left(\mathfrak{f}_1^q + \mathfrak{f}_2^q - \mathfrak{f}_1^q \mathfrak{f}_2^q\right)^{\frac{1}{q}} e^{i2\pi\left(\mathcal{W}_{\mathfrak{f}_1}^q + \mathcal{W}_{\mathfrak{f}_2}^q - \mathcal{W}_{\mathfrak{f}_1}^q \mathcal{W}_{\mathfrak{f}_2}^q\right)^{\frac{1}{q}}}\right) \tag{12}$$

3.

$$\delta P_1 = \left(\left(1 - \left(1 - \mathfrak{t}_1^q\right)^{\delta}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\mathcal{W}_{\mathfrak{t}_1}^q\right)^{\delta}\right)^{\frac{1}{q}}}, \mathfrak{f}_1^{\delta} e^{i2\pi \mathcal{W}_{\mathfrak{f}_1}^{\delta}}\right) \tag{13}$$

4.

$$P_1^{\delta} = \left(\mathfrak{t}_1^{\delta} e^{i2\pi \mathcal{W}_{\mathfrak{t}_1}^{\delta}}, \left(1 - \left(1 - \mathfrak{f}_1^q\right)^{\delta}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_1}^q\right)^{\delta}\right)^{\frac{1}{q}}}\right) \tag{14}$$

Next, we will examine a numerical example for Cq-ROFNs as follows:

Example 1

We consider the two Cq-ROFNs for $q = 2$ and $\lambda = 3$ such that $P_1 = \begin{pmatrix} 0.86e^{i2\pi(0.99)}, \\ 0.10e^{i2\pi(0.03)} \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0.99e^{i2\pi(0.99)}, \\ 0.01e^{i2\pi(0.09)} \end{pmatrix}$, then we have

1. $$P_1 \oplus P_2 = \left(\begin{pmatrix} (0.86)^2 + (0.99)^2 - \\ (0.86)^2(0.99)^2 \end{pmatrix}^{\frac{1}{2}} e^{i2\pi \begin{pmatrix} (0.99)^2 + (0.99)^2 - \\ (0.99)^2(0.99)^2 \end{pmatrix}^{\frac{1}{2}}}, ((0.1)(0.01))\, e^{i2\pi((0.09)(0.03))} \right) = \begin{pmatrix} 0.997e^{i2\pi(0.9998)}, \\ 0.001e^{i2\pi(0.003)} \end{pmatrix}$$

2. $$P_1 \otimes P_2 = \left(((0.86)(0.99))\, e^{i2\pi((0.99)(0.99))}, \begin{pmatrix} (0.1)^2 + (0.01)^2 - \\ (0.1)^2(0.01)^2 \end{pmatrix}^{\frac{1}{2}} e^{i2\pi \begin{pmatrix} (0.03)^2 + (0.09)^2 - \\ (0.03)^2(0.09)^2 \end{pmatrix}^{\frac{1}{2}}} \right) = \begin{pmatrix} 0.85e^{i2\pi(0.001)}, \\ 0.1e^{i2\pi(0.09)} \end{pmatrix}$$

3. $$\lambda P_1 = \left(\left(1 - \left(1 - (0.86)^2\right)^3\right)^{\frac{1}{2}} e^{i2\pi\left(1-\left(1-(0.99)^2\right)^3\right)^{\frac{1}{2}}}, (0.1)^3\, e^{i2\pi(.03)^3} \right) = \begin{pmatrix} 0.99e^{i2\pi(0.99)}, \\ 0.001e^{i2\pi(0.000027)} \end{pmatrix}$$

4. $$P_1^{\lambda} = \left((0.86)^3\, e^{i2\pi(.99)^3}, \left(1 - \left(1 - (0.1)^2\right)^3\right)^{\frac{1}{2}} e^{i2\pi\left(1-\left(1-(.03)^2\right)^3\right)^{\frac{1}{2}}} \right) = \begin{pmatrix} 0.64e^{i2\pi(0.97)}, \\ 0.17e^{i2\pi(0.05)} \end{pmatrix}$$

Definition 9. For a Cq-ROFN $P = \left(\mathfrak{t}e^{i2\pi\mathcal{W}_{\mathfrak{t}}}, \mathfrak{f}e^{i2\pi\mathcal{W}_{\mathfrak{f}}}\right)$, the score and accuracy functions are defined by

$$S(P) = \frac{1}{2}\left(\mathfrak{t}^q - \mathfrak{f}^q + \mathcal{W}_{\mathfrak{t}}^q - \mathcal{W}_{\mathfrak{f}}^q\right) \tag{15}$$

$$H(P) = \frac{1}{2}\left(\mathfrak{t}^q + \mathfrak{f}^q + \mathcal{W}_{\mathfrak{t}}^q + \mathcal{W}_{\mathfrak{f}}^q\right) \tag{16}$$

Definition 10. For any two Cq-ROFNs $P_1 = \left(\mathfrak{t}_1e^{i2\pi\mathcal{W}_{\mathfrak{t}_1}}, \mathfrak{f}_1e^{i2\pi\mathcal{W}_{\mathfrak{f}_1}}\right)$ and $P_2 = \left(\mathfrak{t}_2e^{i2\pi\mathcal{W}_{\mathfrak{t}_2}}, \mathfrak{f}_2e^{i2\pi\mathcal{W}_{\mathfrak{f}_2}}\right)$, then we have the comparison method as follows:

1. If $S(P_1) > S(P_2)$, then $P_1 > P_2$.
2. If $S(P_1) = S(P_2)$, then
    1. If $H(P_1) > H(P_2)$, then If $P_1 > P_2$
    2. If $H(P_1) = H(P_2)$, then If $P_1 = P_2$

Example 2

We consider the two Cq-ROFNs for $q = 2$ and such that $P_1 = \begin{pmatrix} 0.86e^{i2\pi(0.99)}, \\ 0.10e^{i2\pi(0.03)} \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0.99e^{i2\pi(0.99)}, \\ 0.01e^{i2\pi(0.09)} \end{pmatrix}$. Then the score values of the P_1 and P_2 are calculated as follows:

$$\begin{aligned} S(P_1) &= \frac{1}{2}\left(0.86^2 - 0.1^2 + 0.99^2 - 0.03^2\right) \\ &= \frac{1}{2}(0.74 - 0.01 + 0.98 - 0.0009) = 0.86 \\ S(P_1) &= 0.86 \end{aligned}$$

Similarly

$$\begin{aligned} S(P_2) &= \frac{1}{2}\left(0.99^2 - 0.01^2 + 0.99^2 - 0.09^2\right) \\ &= \frac{1}{2}(0.98 - 0.0001 + 0.98 - 0.0081) = 0.98 \\ S(P_2) &= 0.98 \end{aligned}$$

So it is clear that $S(P_2) > S(P_1)$, then we say that $P_2 > P_1$. If $(P_2) = S(P_1)$, the we will use the accuracy function of the Cq-ROFNs.

3. COMPLEX q-RUNG ORTHOPAIR LINGUISTIC SET

Motivated by the notion of Cq-ROFS and LV, we will initiate the novelty of Cq-ROLS by combing the two different concepts. Throughout this article, $\overline{\mathcal{S}}$ is represented the continuous linguistic term set of $\mathcal{S} = \{\mathcal{S}_i / i = 1, 2, .., z\}$.

Definition 11. A Cq-ROLS on a universal fixed set X is given by

$$P = \left\{\left\langle x, \left(\mathcal{S}_{\theta(x)}, \left(\mathfrak{t}'(x), \mathfrak{f}'(x)\right)\right)\right\rangle / x \in X\right\} \tag{17}$$

where $\mathcal{S}_{\theta(x)} \in \overline{\mathcal{S}}$, $\mathfrak{t}'(x) = \mathfrak{t}(x) e^{i2\pi \mathcal{W}_{\mathfrak{t}(x)}}$ and $\mathfrak{f}'(x) = \mathfrak{f}(x) e^{i2\pi \mathcal{W}_{\mathfrak{f}(x)}}$ denoted positive and negative complex-valued degrees respectively, holds $0 \leq \mathfrak{t}^q(x) + \mathfrak{f}^q(x) \leq 1$ and $0 \leq \mathcal{W}^q_{\mathfrak{t}(x)} + \mathcal{W}^q_{\mathfrak{f}(x)} \leq 1, (q \geq 1)$. Further, the refusal degree is defined by: $\mu(x) = (1 - (\mathfrak{t}^q(x) + \mathfrak{f}^q(x)))^{\frac{1}{q}} e^{i2\pi\left(1-\left(\mathcal{W}^q_{\mathfrak{t}(x)}+\mathcal{W}^q_{\mathfrak{f}(x)}\right)\right)^{\frac{1}{q}}}$. Moreover, $\left(\mathcal{S}_{\theta(x)}, \left(\mathfrak{t}(x) e^{i2\pi \mathcal{W}_{\mathfrak{t}(x)}}, \mathfrak{f}(x) e^{i2\pi \mathcal{W}_{\mathfrak{f}(x)}}\right)\right)$ is called complex q-rung orthopair linguistic number (Cq-ROLN). Simply, we write $\left(\mathcal{S}_\theta, \left(\mathfrak{t} e^{i2\pi \mathcal{W}_\mathfrak{t}}, \mathfrak{f} e^{i2\pi \mathcal{W}_\mathfrak{f}}\right)\right)$.

Next, we defined some operations for Cq-ROLNs.

Definition 12. Let $P = \left(\mathcal{S}_\theta, \left(\mathfrak{t} e^{i2\pi \mathcal{W}_\mathfrak{t}}, \mathfrak{f} e^{i2\pi \mathcal{W}_\mathfrak{f}}\right)\right), P_1 = \left(\mathcal{S}_{\theta_1}, \left(\mathfrak{t}_1 e^{i2\pi \mathcal{W}_{\mathfrak{t}_1}}, \mathfrak{f}_1 e^{i2\pi \mathcal{W}_{\mathfrak{f}_1}}\right)\right)$ and $P_2 = \left(\mathcal{S}_{\theta_2}, \left(\mathfrak{t}_2 e^{i2\pi \mathcal{W}_{\mathfrak{t}_2}}, \mathfrak{f}_2 e^{i2\pi \mathcal{W}_{\mathfrak{f}_2}}\right)\right)$ be a three Cq-ROLNs with $\lambda \geq 1$, then we have

1.

$$P_1 \oplus P_2 = \left(\mathcal{S}_{\theta_1+\theta_2}, \left(\left(\mathfrak{t}_1^q + \mathfrak{t}_2^q - \mathfrak{t}_1^q \mathfrak{t}_2^q\right)^{\frac{1}{q}} e^{i2\pi\left(\mathcal{W}^q_{\mathfrak{t}_1}+\mathcal{W}^q_{\mathfrak{t}_2}-\mathcal{W}^q_{\mathfrak{t}_1}\mathcal{W}^q_{\mathfrak{t}_2}\right)^{\frac{1}{q}}}, \left(\mathfrak{f}_1 \mathfrak{f}_2\right) e^{i2\pi\left(\mathcal{W}_{\mathfrak{f}_1}\mathcal{W}_{\mathfrak{f}_2}\right)}\right)\right) \tag{18}$$

2.

$$P_1 \otimes P_2 = \left(\mathcal{S}_{\theta_1\times\theta_2}, \left(\left(\mathfrak{t}_1 \mathfrak{t}_2\right) e^{i2\pi\left(\mathcal{W}_{\mathfrak{t}_1}\mathcal{W}_{\mathfrak{t}_2}\right)}, \left(\mathfrak{f}_1^q + \mathfrak{f}_2^q - \mathfrak{f}_1^q \mathfrak{f}_2^q\right)^{\frac{1}{q}} e^{i2\pi\left(\mathcal{W}^q_{\mathfrak{f}_1}+\mathcal{W}^q_{\mathfrak{f}_2}-\mathcal{W}^q_{\mathfrak{f}_1}\mathcal{W}^q_{\mathfrak{f}_2}\right)^{\frac{1}{q}}}\right)\right) \tag{19}$$

3.

$$\lambda P = \left(\mathcal{S}_{\lambda\times\theta}, \left(\left(1 - \left(1 - \mathfrak{t}^q\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\mathcal{W}^q_\mathfrak{t}\right)^\lambda\right)^{\frac{1}{q}}}, \mathfrak{f}^\lambda e^{i2\pi \mathcal{W}^\lambda_\mathfrak{f}}\right)\right) \tag{20}$$

4.

$$P^\lambda = \left(\mathcal{S}_{\theta\lambda}, \left(\mathfrak{t}^\lambda e^{i2\pi \mathcal{W}^\lambda_\mathfrak{t}}, \left(1 - \left(1 - \mathfrak{f}^q\right)^\lambda\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\mathcal{W}^q_\mathfrak{f}\right)^\lambda\right)^{\frac{1}{q}}}\right)\right) \tag{21}$$

First, we give the numerical example for Cq-ROLNs and the four points of Definition 8. Then, we will proposed the ideas of score function and accuracy function for compression between two Cq-ROFLNs.

Example 3

We will consider the two Cq-ROLNs $P_1 = \left(\mathcal{S}_{4.6}, \left(\begin{matrix} 0.86 e^{i2\pi(0.99)}, \\ 0.10 e^{i2\pi(0.03)} \end{matrix}\right)\right)$ and $P_2 = \left(\mathcal{S}_{2.4}, \left(\begin{matrix} 0.99 e^{i2\pi(0.99)}, \\ 0.01 e^{i2\pi(0.09)} \end{matrix}\right)\right)$ with $q = 2$ and $\lambda = 3$, then we can get

1. $$P_1 \oplus P_2 = \left(\mathcal{S}_{4.6+2.4}, \left(\left(\begin{matrix} (0.86)^2 + (0.99)^2 - \\ (0.86)^2 (0.99)^2 \end{matrix} \right)^{\frac{1}{2}} e^{i2\pi \left(\begin{matrix} (0.99)^2 + (0.99)^2 - \\ (0.99)^2 (0.99)^2 \end{matrix} \right)^{\frac{1}{2}}}, ((0.1)(0.01))\, e^{i2\pi((0.09)(0.03))} \right) \right)$$

$$= \left(\mathcal{S}_7, \left(\begin{matrix} 0.997e^{i2\pi(0.9998)}, \\ 0.001e^{i2\pi(0.003)} \end{matrix} \right) \right),$$

2. $$P_1 \otimes P_2 = \left(\mathcal{S}_{4.6\times 2.4}, \left(((0.86)(0.99))\, e^{i2\pi((0.99)(0.99))}, \left(\begin{matrix} (0.1)^2 + (0.01)^2 - \\ (0.1)^2 (0.01)^2 \end{matrix} \right)^{\frac{1}{2}} e^{i2\pi \left(\begin{matrix} (0.03)^2 + (0.09)^2 - \\ (0.03)^2 (0.09)^2 \end{matrix} \right)^{\frac{1}{2}}} \right) \right)$$

$$= \left(\mathcal{S}_{11.04}, \left(\begin{matrix} 0.85e^{i2\pi(0.001)}, \\ 0.1e^{i2\pi(0.09)} \end{matrix} \right) \right),$$

3. $$\lambda P_1 = \left(\mathcal{S}_{3\times 4.6}, \left(\left(1 - \left(1 - (0.86)^2 \right)^3 \right)^{\frac{1}{2}} e^{i2\pi \left(1 - \left(1 - (0.99)^2 \right)^3 \right)^{\frac{1}{2}}}, (.1)^3\, e^{i2\pi(.03)^3} \right) \right) = \left(\mathcal{S}_{13.8}, \left(\begin{matrix} 0.99e^{i2\pi(0.99)}, \\ 0.001e^{i2\pi(0.000027)} \end{matrix} \right) \right),$$

4. $$P_1^{\lambda} = \left(\mathcal{S}_{(4.6)^3}, \left((0.86)^3\, e^{i2\pi(.99)^3}, \left(1 - \left(1 - (0.1)^2 \right)^3 \right)^{\frac{1}{2}} e^{i2\pi \left(1 - \left(1 - (.03)^2 \right)^3 \right)^{\frac{1}{2}}} \right) \right) = \left(\mathcal{S}_{13.8}, \left(\begin{matrix} 0.64e^{i2\pi(0.97)}, \\ 0.17e^{i2\pi(0.05)} \end{matrix} \right) \right)$$

Definition 13. The score function and accuracy function of Cq-ROLN $P = \left(\mathcal{S}_\theta, \left(\mathfrak{t}e^{i2\pi \mathcal{W}_\mathfrak{t}}, \mathfrak{f}e^{i2\pi \mathcal{W}_\mathfrak{f}} \right) \right)$ are defined as

$$S(P) = \frac{1}{2} \left((\mathfrak{t}^q - \mathfrak{f}^q) + \left(\mathcal{W}_\mathfrak{t}^q - \mathcal{W}_\mathfrak{f}^q \right) \right) \times \theta \tag{22}$$

$$H(P) = \frac{1}{2} \left((\mathfrak{t}^q + \mathfrak{f}^q) + \left(\mathcal{W}_\mathfrak{t}^q + \mathcal{W}_\mathfrak{f}^q \right) \right) \times \theta \tag{23}$$

Definition 14. Let $P_1 = \left(\mathcal{S}_{\theta_1}, \left(\mathfrak{t}_1 e^{i2\pi \mathcal{W}_{\mathfrak{t}1}}, \mathfrak{f}_1 e^{i2\pi \mathcal{W}_{\mathfrak{f}1}} \right) \right)$ and $P_2 = \left(\mathcal{S}_{\theta_2}, \left(\mathfrak{t}_2 e^{i2\pi \mathcal{W}_{\mathfrak{t}2}}, \mathfrak{f}_2 e^{i2\pi \mathcal{W}_{\mathfrak{f}2}} \right) \right)$ be a two Cq-ROLNs, then

1. If $S(P_1) > S(P_2)$, then $P_1 > P_2$.
2. If $S(P_1) = S(P_2)$, then
    1. If $H(P_1) > H(P_2)$, then $P_1 > P_2$.
    2. If $H(P_1) = H(P_2)$, then $P_1 = P_2$.

Example 4

We will consider the two Cq-ROLNs $P_1 = \left(\mathcal{S}_{4.6}, \left(\begin{matrix} 0.86e^{i2\pi(0.99)}, \\ 0.10e^{i2\pi(0.03)} \end{matrix} \right) \right)$ and $P_2 = \left(\mathcal{S}_{2.4}, \left(\begin{matrix} 0.99e^{i2\pi(0.99)}, \\ 0.01e^{i2\pi(0.09)} \end{matrix} \right) \right)$ for $q = 2$. Then the score values of the P_1 and P_2 are calculated as

$$S(P_1) = \frac{1}{2} \left(\left(0.86^2 - .1^2 \right) + \left(0.99^2 - 0.03^2 \right) \right) \times 4.6 = 0.86 \times 4.6$$
$$S(P_1) = 3.86$$

Similarly

$$S(P_2) = \frac{1}{2} \left(\left(0.99^2 - .01^2 \right) + \left(0.99^2 - 0.09^2 \right) \right) \times 2.4 = 0.98 \times 4.6$$
$$S(P_2) = 2.55$$

So it is clear that $S(P_1) > S(P_2)$, then we say that $P_1 > P_2$. If $S(P_2) = S(P_1)$, then we will use the accuracy function of the Cq-ROFLNs.

4. COMPLEX q-RUNG ORTHOPAIR LINGUISTIC HM OPERATORS

In this section, we generalize the HM operator to Cq-ROLS and propose the concepts of Cq-ROLHM, Cq-ROLWHM, Cq-ROLGHM, Cq-ROLWGHM operators and discuss their properties in detailed, where s,t.

4.1. Complex q-Rung Orthopair Linguistic Heronian Mean (Cq-Rolhm) Operators

Definition 15. Let $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), (i = 1, 2, \ldots, n)$ be a family of Cq-ROLNs, then the $Cq-ROLHM^{s,t}$ is defined: $Cq-ROLHM^{s,t}: \xi^n \to \xi$ by

$$Cq-ROLHM^{s,t}(P_1, P_2, .., P_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=I}^{n} P_i^s P_j^t\right)^{\frac{1}{s+t}} \tag{24}$$

where ξ^n denotes the family of all Cq-ROLNs.

According to the operational laws of Cq-ROLNs, we can get the following results.

Theorem 1. *Let $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), (i = 1, 2, \ldots, n)$ be a family of Cq-ROLNs, we can get*

$$\begin{aligned} &Cq-ROLHM^{s,t}(P_1, P_2, .., P_n) = \\ &\left(\mathcal{S}_{\left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=I}^{n} \theta_i^s \theta_j^t\right)^{\frac{1}{s+t}}}, \left(\left(1 - \prod_{i=1}^{n} \prod_{j=I}^{n} \left(1 - \mathfrak{t}_i^{sq} \mathfrak{t}_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi \left(1 - \prod_{i=1}^{n} \prod_{j=I}^{n} \left(1 - \mathcal{W}_{\mathfrak{t}_i}^{sq} \mathcal{W}_{\mathfrak{t}_j}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}, \right.\right. \\ &\left.\left. \left(1 - \left(1 - \prod_{i=1}^{n} \prod_{j=I}^{n} \left(1 - \left(1 - \mathfrak{f}_i^q\right)^s \left(1 - \mathfrak{f}_j^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi \left(1 - \left(1 - \prod_{i=1}^{n} \prod_{j=I}^{n} \left(1 - \left(1 - \mathcal{W}_{\mathfrak{f}_i}^q\right)^s \left(1 - \mathcal{W}_{\mathfrak{f}_j}^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}} \right)\right) \end{aligned} \tag{25}$$

Proof: Using the Definition 12, we get

$$P_i^s = \left(\mathcal{S}_{\theta_i^s}, \left(\mathfrak{t}_i^s e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}^s}, \left(1 - \left(1 - \mathfrak{f}_i^q\right)^s\right)^{\frac{1}{q}} e^{i2\pi \left(1 - \left(1 - \mathcal{W}_{\mathfrak{f}_i}^q\right)^s\right)^{\frac{1}{q}}}\right)\right),$$

$$P_j^t = \left(\mathcal{S}_{\theta_j^t}, \left(\mathfrak{t}_j^t e^{i2\pi \mathcal{W}_{\mathfrak{t}_j}^t}, \left(1 - \left(1 - \mathfrak{f}_j^q\right)^t\right)^{\frac{1}{q}} e^{i2\pi \left(1 - \left(1 - \mathcal{W}_{\mathfrak{f}_j}^q\right)^t\right)^{\frac{1}{q}}}\right)\right),$$

$$P_i^s P_j^t = \left(\mathcal{S}_{\theta_i^s \theta_j^t}, \left(\mathfrak{t}_i^s \mathfrak{t}_j^t e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}^s \mathcal{W}_{\mathfrak{t}_j}^t}, \left(1 - \left(1 - \mathfrak{f}_i^q\right)^s \left(1 - \mathfrak{f}_j^q\right)^t\right) e^{i2\pi \left(1 - \left(1 - \mathcal{W}_{\mathfrak{f}_i}^q\right)^s \left(1 - \mathcal{W}_{\mathfrak{f}_j}^q\right)^t\right)}\right)\right),$$

$$\sum_{j=1}^{n} P_i^s P_j^t = \left(\mathcal{S}_{\sum_{j=1}^{n} \theta_i^s \theta_j^t}, \left(\begin{array}{c} \left(1-\prod_{j=1}^{n}\left(1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\prod_{j=1}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}^{sq}\mathcal{W}_{\mathfrak{t}_j}^{tq}\right)\right)^{\frac{1}{q}}}, \\ \prod_{j=1}^{n}\left(1-\left(1-\mathfrak{f}_i^{q}\right)^{s}\left(1-\mathfrak{f}_j^{q}\right)^{t}\right) e^{i2\pi\prod_{j=1}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^{q}\right)^{s}\left(1-\mathcal{W}_{\mathfrak{f}_j}^{q}\right)^{t}\right)} \end{array} \right) \right),$$

then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P_i^s P_j^t = \left(\mathcal{S}_{\sum_{i=1}^{n}\sum_{j=1}^{n} \theta_i^s \theta_j^t}, \left(\begin{array}{c} \left(1-\prod_{i=1}^{n}\left(\prod_{j=I}^{n}\left(1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq}\right)\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\prod_{i=1}^{n}\left(\prod_{j=I}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}^{sq}\mathcal{W}_{\mathfrak{t}_j}^{tq}\right)\right)\right)^{\frac{1}{q}}}, \\ \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i^{q}\right)^{s}\left(1-\mathfrak{f}_j^{q}\right)^{t}\right) e^{i2\pi\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^{q}\right)^{s}\left(1-\mathcal{W}_{\mathfrak{f}_j}^{q}\right)^{t}\right)} \end{array} \right) \right),$$

$$\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n} P_i^s P_j^t = .$$

$$\left(\mathcal{S}_{\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n} \theta_i^s \theta_j^t}, \left(\begin{array}{c} \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq}\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}^{sq}\mathcal{W}_{\mathfrak{t}_j}^{tq}\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q}}}, \\ \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i^{q}\right)^{s}\left(1-\mathfrak{f}_j^{q}\right)^{t}\right)\right)^{\frac{2}{n(n+1)}} e^{i2\pi\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^{q}\right)^{s}\left(1-\mathcal{W}_{\mathfrak{f}_j}^{q}\right)^{t}\right)\right)^{\frac{2}{n(n+1)}}} \end{array} \right) \right)$$

So, we obtained

$$Cq-ROLHM^{s,t}(P_1,P_2,..,P_n) = \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n} P_i^s P_j^t\right)^{\frac{1}{s+t}} =$$

$$\left(\begin{array}{c} \mathcal{S}_{\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\theta_i^s\theta_j^t\right)^{\frac{1}{s+t}}}, \\ \left(\begin{array}{c} \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}^{sq}\mathcal{W}_{\mathfrak{t}_j}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}, \\ \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i^{q}\right)^{s}\left(1-\mathfrak{f}_j^{q}\right)^{t}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^{q}\right)^{s}\left(1-\mathcal{W}_{\mathfrak{f}_j}^{q}\right)^{t}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}} \end{array} \right) \end{array} \right)$$

Further, we discuss the properties of Cq-ROLNs as follows.

Theorem 2. *(Monotonicity) Let* $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi\mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi\mathcal{W}_{\mathfrak{f}_i}}\right)\right)$ *and* $P_i' = \left(\mathcal{S}_{\theta_i'}, \left(\mathfrak{t}_i' e^{i2\pi\mathcal{W}_{\mathfrak{t}_i}'}, \mathfrak{f}_i' e^{i2\pi\mathcal{W}_{\mathfrak{f}_i}'}\right)\right)$, $(i = 1, 2, \ldots, n)$ *be two families of Cq-ROLNs, if* $P_i \le P_i' \Leftrightarrow \theta_i \le \theta_i', \mathfrak{t}_i \le \mathfrak{t}_i', \mathcal{W}_{\mathfrak{t}_i} \le \mathcal{W}_{\mathfrak{t}_i}'$ *and* $\mathfrak{f}_i \le \mathfrak{f}_i', \mathcal{W}_{\mathfrak{f}_i} \le \mathcal{W}_{\mathfrak{f}_i}'$ *for all* $i = 1, 2, \ldots, n$. *Then*

$$Cq-ROLHM^{s,t}(P_1,P_2,..,P_n) \le Cq-ROLHM^{s,t}(P_1,P_2,..,P_n)$$

Proof: Since $P_i \le P_i' \Leftrightarrow \theta_i \le \theta_i', \mathfrak{t}_i \le \mathfrak{t}_i', \mathcal{W}_{\mathfrak{t}_i} \le \mathcal{W}_{\mathfrak{t}_i}'$ and $\mathfrak{f}_i \ge \mathfrak{f}_i', \mathcal{W}_{\mathfrak{f}_i} \ge \mathcal{W}_{\mathfrak{f}_i}'$ and $P_j \le P_j' \Leftrightarrow \theta_j \le \theta_j', \mathfrak{t}_j \le \mathfrak{t}_j', \mathcal{W}_{\mathfrak{t}_j} \le \mathcal{W}_{\mathfrak{t}_j}'$ and $\mathfrak{f}_j \ge \mathfrak{f}_j', \mathcal{W}_{\mathfrak{f}_j} \ge \mathcal{W}_{\mathfrak{f}_j}'$ for all $i = 1, 2, \ldots, n$ and $j = i, i+1, .., n$. Then it is clear that for linguistic number

$$\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\theta_i^s\theta_j^t\right)^{\frac{1}{s+t}} \le \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\theta_i'^{s}\theta_j'^{t}\right)^{\frac{1}{s+t}}$$

Nest we will check the real-valued membership grade such that $\mathfrak{t}_i \le \mathfrak{t}_i', \mathcal{W}_{\mathfrak{t}_i} \le \mathcal{W}_{\mathfrak{t}_i}'$ and $\mathfrak{t}_j \le \mathfrak{t}_j', \mathcal{W}_{\mathfrak{t}_j} \le \mathcal{W}_{\mathfrak{t}_j}'$, then

$$\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq} \le \mathfrak{t}_i'^{sq}\mathfrak{t}_j'^{tq} \Rightarrow 1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq} \ge 1-\mathfrak{t}_i'^{sq}\mathfrak{t}_j'^{tq} \Rightarrow \left(1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq}\right)^{\frac{1}{n(n+1)}} \ge \left(1-\mathfrak{t}_i'^{sq}\mathfrak{t}_j'^{tq}\right)^{\frac{1}{n(n+1)}}$$

$$\Rightarrow \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq}\right)^{\frac{2}{n(n+1)}} \ge \prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathfrak{t}_i'^{sq}\mathfrak{t}_j'^{tq}\right)^{\frac{1}{n(n+1)}}$$

$$\Rightarrow \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} \le \left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathfrak{t}_i'^{sq}\mathfrak{t}_j'^{tq}\right)^{\frac{1}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}$$

Similarly procedure for imaginary-valued membership grades, we get

$$\Rightarrow \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}^{sq}\mathcal{W}_{\mathfrak{t}_j}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} \le \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}'^{sq}\mathcal{W}_{\mathfrak{t}_j}'^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}$$

Combined both values, we have

$$\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}^{sq}\mathcal{W}_{\mathfrak{t}_j}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}$$

$$\le \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathfrak{t}_i'^{sq}\mathfrak{t}_j'^{tq}\right)^{\frac{1}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}'^{sq}\mathcal{W}_{\mathfrak{t}_j}'^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}$$

Nest we will describe the real-valued non-membership grade such that $\mathfrak{f}_i \ge \mathfrak{f}_i'$, $\mathcal{W}_{\mathfrak{f}_i} \ge \mathcal{W}_{\mathfrak{f}_i}'$ and $\mathfrak{f}_j \ge \mathfrak{f}_j'$, $\mathcal{W}_{\mathfrak{f}_j} \ge \mathcal{W}_{\mathfrak{f}_j}'$. Then

$$\mathfrak{f}_i^q\mathfrak{f}_j^q \ge \mathfrak{f}_i'^q\mathfrak{f}_j'^q \Rightarrow \left(1-\mathfrak{f}_i^q\right)^s\left(1-\mathfrak{f}_j^q\right)^t \ge \left(1-\mathfrak{f}_i'^q\right)^s\left(1-\mathfrak{f}_j'^q\right)^t$$

$$\Rightarrow \left(1-\left(1-\mathfrak{f}_i^q\right)^s\left(1-\mathfrak{f}_j^q\right)^t\right)^{\frac{2q}{n(n+1)}} \le \left(1-\left(1-\mathfrak{f}_i'^q\right)^s\left(1-\mathfrak{f}_j'^q\right)^t\right)^{\frac{2q}{n(n+1)}}$$

$$\Rightarrow \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i^q\right)^s\left(1-\mathfrak{f}_j^q\right)^t\right)^{\frac{2q}{n(n+1)}} \le \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i'^q\right)^s\left(1-\mathfrak{f}_j'^q\right)^t\right)^{\frac{2q}{n(n+1)}}$$

$$\Rightarrow 1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i^q\right)^s\left(1-\mathfrak{f}_j^q\right)^t\right)^{\frac{2q}{n(n+1)}} \ge 1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i'^q\right)^s\left(1-\mathfrak{f}_j'^q\right)^t\right)^{\frac{2q}{n(n+1)}}$$

$$\Rightarrow \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i^q\right)^s\left(1-\mathfrak{f}_j^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}} \ge \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i'^q\right)^s\left(1-\mathfrak{f}_j'^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}$$

$$\Rightarrow 1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i^q\right)^s\left(1-\mathfrak{f}_j^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}} \le 1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i'^q\right)^s\left(1-\mathfrak{f}_j'^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}$$

$$\Rightarrow \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i^{q}\right)^{s}\left(1-\mathfrak{f}_j^{q}\right)^{t}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} \leq \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i'^{q}\right)^{s}\left(1-\mathfrak{f}_j'^{q}\right)^{t}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}$$

Similarly procedure for imaginary-valued non-membership grades, we get

$$\Rightarrow \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^{q}\right)^{s}\left(1-\mathcal{W}_{\mathfrak{f}_j}^{q}\right)^{t}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} \leq \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}'^{Q}\right)^{s}\left(1-\mathcal{W}_{\mathfrak{f}_j}'^{Q}\right)^{t}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}$$

Combined both values, we have

$$\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i^{q}\right)^{s}\left(1-\mathfrak{f}_j^{q}\right)^{t}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^{q}\right)^{s}\left(1-\mathcal{W}_{\mathfrak{f}_j}^{q}\right)^{t}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}}$$

So by combining the values of complex-valued membership and complex-valued non-membership grades, then we get

$$\left(\begin{array}{l}\mathcal{S}\\ \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\theta_i^{s}\theta_j^{t}\right)^{\frac{1}{s+t}}, \\ \left(\begin{array}{l}\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}^{sq}\mathcal{W}_{\mathfrak{t}_j}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}, \\ \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i^{q}\right)^{s}\left(1-\mathfrak{f}_j^{q}\right)^{t}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^{q}\right)^{s}\left(1-\mathcal{W}_{\mathfrak{f}_j}^{q}\right)^{t}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}}\end{array}\right)\end{array}\right)$$

$$\leq \left(\begin{array}{l}\mathcal{S}\\ \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\theta_i'^{s}\theta_j'^{t}\right)^{\frac{1}{s+t}}, \\ \left(\begin{array}{l}\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathfrak{t}_i'^{sq}\mathfrak{t}_j'^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}'^{sq}\mathcal{W}_{\mathfrak{t}_j}'^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}, \\ \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i'^{q}\right)^{s}\left(1-\mathfrak{f}_j'^{q}\right)^{t}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}'^{q}\right)^{s}\left(1-\mathcal{W}_{\mathfrak{f}_j}'^{q}\right)^{t}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}}\end{array}\right)\end{array}\right)$$

$$\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}P_i^sP_j^t\right)^{\frac{1}{s+t}} \le \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}Q_i^sQ_j^t\right)^{\frac{1}{s+t}},$$

i.e., $Cq-ROLHM^{s,t}(P_1,P_2,..,P_n) \le Cq-ROLHM^{s,t}(Q_1,Q_2,..,Q_n)$

Hence proved the result.

Theorem 3. *(Idempotency) Let* $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), (i=1,2,\ldots,n)$ *be a family of Cq-ROLNs, if* $P_i = P$ *for all* $i = 1,2,\ldots,n$. *Then*

$$Cq-ROLHM^{s,t}(P_1,P_2,..,P_n) = P$$

Proof: If $P_i = P$, for all $i = 1,2,\ldots,n$, then

$$Cq-ROLHM^{s,t}(P_1,P_2,..,P_n) = \left(\begin{array}{l}\mathcal{S}_{\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\theta_i^s\theta_j^t\right)^{\frac{1}{s+t}}}, \\ \left(\begin{array}{l}\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}^{sq}\mathcal{W}_{\mathfrak{t}_j}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}, \\ \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}_i^q\right)^s\left(1-\mathfrak{f}_j^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^q\right)^s\left(1-\mathcal{W}_{\mathfrak{f}_j}^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}}\end{array}\right)\end{array}\right)$$

$$= \left(\begin{array}{l}\mathcal{S}_{\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\theta^s\theta^t\right)^{\frac{1}{s+t}}}, \\ \left(\begin{array}{l}\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathfrak{t}^{sq}\mathfrak{t}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\mathcal{W}_{\mathfrak{t}}^{sq}\mathcal{W}_{\mathfrak{t}}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}, \\ \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathfrak{f}^q\right)^s\left(1-\mathfrak{f}^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}}^q\right)^s\left(1-\mathcal{W}_{\mathfrak{f}}^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}}\end{array}\right)\end{array}\right)$$

$$= \left(\begin{array}{l} \mathcal{S}_{(\theta^s\theta^t)^{\frac{1}{s+t}}}, \\ \left(\begin{array}{l} \left(\mathfrak{t}^s\mathfrak{t}^t\right)^{\frac{1}{(s+t)}} e^{i2\pi\left(\mathcal{W}_{\mathfrak{t}}^s\mathcal{W}_{\mathfrak{t}}^t\right)^{\frac{1}{(s+t)}}}, \\ \left(1-\left(1-\mathfrak{f}^q\right)^s\left(1-\mathfrak{f}^q\right)^t\right)^{\frac{1}{s+t}} e^{i2\pi\left(1-\left(1-\mathcal{W}_{\mathfrak{f}}^q\right)^s\left(1-\mathcal{W}_{\mathfrak{f}}^q\right)^t\right)^{\frac{1}{s+t}}} \end{array} \right) \end{array} \right)$$

$$= \left(P^sP^t\right)^{\frac{1}{s+t}}$$

$$= \left(P^{s+t}\right)^{\frac{1}{s+t}} = P$$

Hence proved the result.

Theorem 4. *(Boundedness) The Cq-ROLHM operator lies between the max and min operators*

$$min\left(P_1, P_2, .., P_n\right) \le Cq - ROLHM^{s,t}\left(P_1, P_2, .., P_n\right) \le max\left(P_1, P_2, .., P_n\right)$$

Proof: When, we consider $a = min\left(P_1, P_2, .., P_n\right)$ and $b = max\left(P_1, P_2, .., P_n\right)$, then using the result of monotonicity, we have

$$min\left(a, a, a, .., a\right) \le Cq - ROLHM^{s,t}\left(P_1, P_2, .., P_n\right) \le max\left(b, b, b, .., b\right)$$

Moreover, $min\left(a, a, a, .., a\right) = a$ and $max\left(b, b, b, .., b\right) = b$, then

$$a \le Cq - ROLHM^{s,t}\left(P_1, P_2, .., P_n\right) \le b$$

That is

$$min\left(P_1, P_2, .., P_n\right) \le Cq - ROLHM^{s,t}\left(P_1, P_2, .., P_n\right) \le max\left(P_1, P_2, .., P_n\right)$$

Hence completed the result.

4.2. Special Cases

In this sub-section, the particular cases of Cq-ROLHM operator is discuss about the parameters s and t.

1. When $t \to 0$, we have

$$Cq - ROLHM^{s,0}\left(P_1, P_2, .., P_n\right) =$$

$$\lim_{t\to 0} \left(\begin{array}{l} \mathcal{S}_{\left(\frac{2}{n(n+1)}\sum_{i=1}^n\sum_{j=i}^n \theta_i^s\theta_j^t\right)^{\frac{1}{s+t}}}, \\ \left(\begin{array}{l} \left(1-\prod_{i=1}^n\prod_{j=i}^n\left(1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^n\prod_{j=1}^n\left(1-\mathcal{W}_{\mathfrak{t}_i}^{sq}\mathcal{W}_{\mathfrak{t}_j}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}, \\ \left(1-\left(1-\prod_{i=1}^n\prod_{j=i}^n\left(1-\left(1-\mathfrak{f}_i^q\right)^s\left(1-\mathfrak{f}_j^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^n\prod_{j=1}^n\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^q\right)^s\left(1-\mathcal{W}_{\mathfrak{f}_j}^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}} \end{array} \right) \end{array} \right)$$

$$= \left(\begin{array}{l} \mathcal{S}_{\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}(n+1-i)\theta_i^s\right)^{\frac{1}{s}}}, \\ \left(\left(1-\left(\prod_{i=1}^{n}\left(1-\mathfrak{t}_i^{sq}\right)^{(n+1-i)}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{qs}} e^{i2\pi\left(1-\left(\prod_{i=1}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}^{sq}\right)^{(n+1-i)}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{qs}}}, \right. \\ \left. \left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\mathfrak{f}_i^q\right)^s\right)^{(n+1-i)}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^q\right)^s\right)^{(n+1-i)}\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s}}\right)^{\frac{1}{q}}} \right) \end{array} \right),$$

and it is called q-rung orthopair linguistic generalized linear descending weighted mean (q-ROLGLDWM) operator.

2. When $s \to 0$, then

$$Cq-ROLHM^{0,t}\left(P_1,P_2,..,P_n\right) =$$

$$\lim_{s\to 0} \left(\begin{array}{l} \mathcal{S}_{\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_i^s\theta_j^t\right)^{\frac{1}{s+t}}}, \\ \left(\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathfrak{t}_i^{sq}\mathfrak{t}_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}^{sq}\mathcal{W}_{\mathfrak{t}_j}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}, \right. \\ \left. \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{f}_i^q\right)^s\left(1-\mathfrak{f}_j^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^q\right)^s\left(1-\mathcal{W}_{\mathfrak{f}_j}^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}} \right) \end{array} \right)$$

$$= \left(\begin{array}{l} \mathcal{S}_{\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}i\theta_i^t\right)^{\frac{1}{t}}}, \\ \left(\left(1-\left(\prod_{i=1}^{n}\left(1-\mathfrak{t}_i^{tq}\right)^i\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{qt}} e^{i2\pi\left(1-\left(\prod_{i=1}^{n}\left(1-\mathcal{W}_{\mathfrak{t}_i}^{tq}\right)^i\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{qt}}}, \right. \\ \left. \left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\mathfrak{f}_i^q\right)^t\right)^i\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^q\right)^t\right)^i\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{t}}\right)^{\frac{1}{q}}} \right) \end{array} \right),$$

and it is called q-rung orthopair linguistic generalized linear ascending weighted mean (q-ROLGLAWM) operator.

3. When $s = t = 1$, then

$$Cq-ROLHM^{1,1}(P_1, P_2, .., P_n) = \left(\mathcal{S}_{\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_i^1\theta_j^1\right)^{\frac{1}{2}}}, \left(\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathfrak{t}_i^q\mathfrak{t}_j^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{2q}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-w_{\mathfrak{t}_i}^q w_{\mathfrak{t}_j}^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{2q}}}, \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{f}_i^q\right)^1\left(1-\mathfrak{f}_j^q\right)^1\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{2}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-w_{\mathfrak{f}_i}^q\right)^1\left(1-w_{\mathfrak{f}_j}^q\right)^1\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{2}}\right)^{\frac{1}{q}}} \right) \right)$$

is called q-rung orthopair linguistic line Heronian mean (q-ROLLHM) operator.

4. When $q = 2$, then

$$Cq-ROLHM^{s,t}(P_1, P_2, .., P_n) = \left(\mathcal{S}_{\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_i^s\theta_j^t\right)^{\frac{1}{s+t}}}, \left(\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathfrak{t}_i^{2s}\mathfrak{t}_j^{2t}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{2(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-w_{\mathfrak{t}_i}^{2s} w_{\mathfrak{t}_j}^{2t}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{2(s+t)}}}, \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{f}_i^2\right)^s\left(1-\mathfrak{f}_j^2\right)^t\right)^{\frac{4}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{2}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-w_{\mathfrak{f}_i}^2\right)^s\left(1-w_{\mathfrak{f}_j}^2\right)^t\right)^{\frac{4}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{2}}} \right) \right)$$

is called pythagorean linguistic Heromian mean (PLHM) operator.

5. When $q = 1$, then

$$Cq-ROLHM^{s,t}(P_1, P_2, .., P_n) = \left(\mathcal{S}_{\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_i^s\theta_j^t\right)^{\frac{1}{s+t}}}, \left(\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathfrak{t}_i^{s}\mathfrak{t}_j^{t}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-w_{\mathfrak{t}_i}^{s} w_{\mathfrak{t}_j}^{t}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{(s+t)}}}, \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{f}_i^1\right)^s\left(1-\mathfrak{f}_j^1\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{1} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-w_{\mathfrak{f}_i}^1\right)^s\left(1-w_{\mathfrak{f}_j}^1\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{1}} \right) \right)$$

is called intuitionistic linguistic Heromian mean (ILHM) operator.

4.3. Complex q-Rung Orthopair Linguistic Weighted Heronian Mean (Cq-Rolwhm) Operators

Last sub-section, we proposed the Cq-ROFHM operator without weight vectors. Therefore, we propose the weighted form of Cq-ROFHM called Cq-ROFWHM operator. Further, $\omega = (\omega_1, \omega_2, .., \omega_n)^T$ represented the weight vectors with $\sum_{i=1}^{n} \omega_i = 1$.

Definition 16. Let $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), i = 1, 2, \ldots, n$ be a family of Cq-ROLNs, then the $Cq - ROLWHM^{s,t}$ is defined: $Cq - ROLWHM^{s,t} : \xi^n \to \xi$ by

$$Cq - ROLWHM^{s,t}(P_1, P_2, .., P_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (n\omega_i P_i)^s \left(n\omega_j P_j\right)^t\right)^{\frac{1}{s+t}} \tag{26}$$

where ξ^n is denoted the family of all Cq-ROLNs.

According to the operational laws of Cq-ROLNs, we can get the following result.

Theorem 5. *Let $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), (i = 1, 2, \ldots, n)$ be a family of Cq-ROLNs. According the Definition 16 and the operational laws of Cq-ROLNs, we have*

$$Cq - ROLWHM^{s,t}(P_1, P_2, .., P_n) = \tag{27}$$

$$\left(\begin{array}{l} \mathcal{S}_{\left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (n\omega_i \theta_i)^s \left(n\omega_j \theta_j\right)^t\right)^{\frac{1}{s+t}}}, \\ \left(\begin{array}{l} \left(1 - \prod_{i=1}^{n} \prod_{j=1}^{n} \left(1 - \left(1 - \left(1 - \mathfrak{t}_i^q\right)^{n\omega_i}\right)^{\frac{2s}{n(n+1)}} \left(1 - \left(1 - \mathfrak{t}_j^q\right)^{n\omega_j}\right)^{\frac{2t}{n(n+1)}}\right)\right)^{\frac{1}{q(s+t)}} e^{i2\pi \left(1 - \prod_{i=1}^{n} \prod_{j=1}^{n} \left(1 - \left(1 - \left(1 - \mathcal{W}_{\mathfrak{t}_i}^q\right)^{n\omega_i}\right)^{\frac{2s}{n(n+1)}} \left(1 - \left(1 - \mathcal{W}_{\mathfrak{t}_j}^q\right)^{n\omega_j}\right)^{\frac{2t}{n(n+1)}}\right)\right)^{\frac{1}{q(s+t)}}}, \\ \left(1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1}^{n} \left(1 - \left(1 - \mathfrak{f}_i^{n\omega_i q}\right)^s \left(1 - \mathfrak{f}_j^{n\omega_j q}\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi \left(1 - \left(1 - \prod_{i=1}^{n} \prod_{j=1}^{n} \left(1 - \left(1 - \mathcal{W}_{\mathfrak{f}_i}^{n\omega_i q}\right)^s \left(1 - \mathcal{W}_{\mathfrak{f}_j}^{n\omega_j q}\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}} \end{array}\right) \end{array}\right)$$

Proof: Similar to Theorem 1.

Further, we explore some properties of Cq-ROLNs as follows:

Theorem 6. *(Monotonicity) Let $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right)$ and $Q_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), (i = 1, 2, \ldots, n)$ be two families of Cq-ROLNs, if $P_i \leq Q_i$ for all $i = 1, 2, \ldots, n$. Then*

$$Cq - ROLWHM^{s,t}(P_1, P_2, .., P_n) \leq Cq - ROLWHM^{s,t}(Q_1, Q_2, .., Q_n)$$

Proof: Similar to Theorem 2.

Theorem 7. *(Idempotency) Let* $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), (i = 1, 2, \ldots, n)$ *be a family of Cq-ROLNs, if* $P_i = P$ *for all* $i = 1, 2, \ldots, n$*. Then*

$$Cq - ROLWHM^{s,t}\left(P_1, P_2, .., P_n\right) = P$$

Proof: Similar to Theorem 3.

Theorem 8. *(Boundedness) The Cq-ROLWHM operator lies between the max and min operators*

$$min\left(P_1, P_2, .., P_n\right) \leq Cq - ROLWHM^{s,t}\left(P_1, P_2, .., P_n\right) \leq max\left(P_1, P_2, .., P_n\right)$$

Proof: Similar to Theorem 3.

4.4. Complex q-Rung Orthopair Linguistic Geometric Heronian Mean (Cq-Rolghm) Operators

Definition 17. Let $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), i = 1, 2, \ldots, n$ be a family of Cq-ROLNs, then the $Cq - ROLGHM^{s,t}$ is defined: $Cq - ROLGHM^{s,t} : \xi^n \rightarrow \xi$ by

$$Cq - ROLGHM^{s,t}\left(P_1, P_2, .., P_n\right) = \left(\frac{1}{s+t}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(sP_i + tP_j\right)^{\frac{2}{n(n+1)}}\right) \tag{28}$$

where ξ^n is denoted the family of all Cq-ROLNs.

According to the operational laws of Cq-ROLNs, we can get the following result.

Theorem 9. *Let* $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), (i = 1, 2, \ldots, n)$ *be a family of Cq-ROLNs, we can get*

$$Cq - ROLGHM^{s,t}\left(P_1, P_2, .., P_n\right) = \tag{29}$$
$$\left(\begin{array}{l}\mathcal{S}_{\left(\frac{1}{s+t}\prod_{i=1}^{n}\prod_{j=1}^{n}\left(s\theta_i + t\theta_j\right)^{\frac{2}{n(n+1)}}\right)}, \\ \left(\begin{array}{l}\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{t}_i^q\right)^s\left(1-\mathfrak{t}_j^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{t}_i}^q\right)^s\left(1-\mathcal{W}_{\mathfrak{t}_j}^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}}, \\ \left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathfrak{f}_i^{sq}\mathfrak{f}_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathcal{W}_{\mathfrak{f}_i}^{sq}\mathcal{W}_{\mathfrak{f}_j}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}\end{array}\right)\end{array}\right)$$

Proof: Similar to Theorem 1.

Further, we discuss some properties of Cq-ROLNs as follows:

Theorem 10. *(Monotonicity) Let* $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right)$ *and* $Q_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), (i = 1, 2, \ldots, n)$ *be two families of Cq-ROLNs, if* $P_i \leq Q_i$ *for all* $i = 1, 2, \ldots, n$*. Then*

$$Cq - ROLGHM^{s,t}\left(P_1, P_2, .., P_n\right) \leq Cq - ROLGHM^{s,t}\left(Q_1, Q_2, .., Q_n\right)$$

Proof: Similar to Theorem 2.

Theorem 11. *(Idempotency) Let* $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), (i = 1, 2, \ldots, n)$ *be a family of Cq-ROLNs, if* $P_i = P$ *for all* $i = 1, 2, \ldots, n$. *Then*

$$Cq-ROLGHM^{s,t}\left(P_1, P_2, .., P_n\right) = P$$

Proof: Similar to Theorem 3.

Theorem 12. *(Boundedness) The Cq-ROLWHM operator lies between the max and min operators*

$$min\left(P_1, P_2, .., P_n\right) \leq Cq-ROLGHM^{s,t}\left(P_1, P_2, .., P_n\right) \leq max\left(P_1, P_2, .., P_n\right)$$

Proof: Similar to Theorem 4.

4.5. Special Cases

In this sub-section, the particular cases of Cq-ROLGHM operator is discussed based on the parameters s and t.

1. When $t \to 0$, then

$$Cq-ROLGHM^{s,0}\left(P_1, P_2, .., P_n\right) =$$

$$\lim_{t\to 0}\left(\begin{array}{l}\mathcal{S}_{\left(\frac{1}{s+t}\prod_{i=1}^{n}\prod_{j=1}^{n}\left(s\theta_i + t\theta_j\right)^{\frac{2}{n(n+1)}}\right)},\\ \left(\begin{array}{l}\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{t}_i^q\right)^s\left(1-\mathfrak{t}_j^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{t}_i}^q\right)^s\left(1-\mathcal{W}_{\mathfrak{t}_j}^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}},\\ \left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathfrak{f}_i^{sq}\mathfrak{f}_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathcal{W}_{\mathfrak{f}_i}^{sq}\mathcal{W}_{\mathfrak{f}_j}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}\end{array}\right)\end{array}\right)$$

$$=\left(\begin{array}{l}\mathcal{S}_{\left(\frac{1}{s}\left(\prod_{j=i}^{n}\left(s\theta_i\right)^{n+1-i}\right)^{\frac{2}{n(n+1)}}\right)},\\ \left(\begin{array}{l}\left(1-\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{t}_i^q\right)^s\right)^{n+1-i}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\left(\prod_{j=1}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{t}_i}^q\right)^s\right)^{n+1-i}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}},\\ \left(1-\left(\prod_{j=1}^{n}\left(1-\mathfrak{f}_i^{sq}\right)^{n+1-i}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\left(\prod_{j=1}^{n}\left(1-\mathcal{W}_{\mathfrak{f}_i}^{sq}\right)^{n+1-i}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}}\end{array}\right)\end{array}\right)$$

is called q-rung orthopair linguistic generalized geometric linear descending weighted mean (q-ROLGGLDWM) operator.

2. When $t \to 0$, then

$$Cq-ROLGHM^{0,t}(P_1,P_2,..,P_n)=$$

$$\lim_{s\to 0}\left(\begin{array}{l} S_{\left(\frac{1}{s+t}\prod_{i=1}^{n}\prod_{j=1}^{n}\left(s\theta_i+t\theta_j\right)^{\frac{2}{n(n+1)}}\right)}, \\ \left(\begin{array}{l} \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{t}_i^q\right)^s\left(1-\mathfrak{t}_j^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-w_{\mathfrak{t}_i}^q\right)^s\left(1-w_{\mathfrak{t}_j}^q\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}}, \\ \left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathfrak{f}_i^{sq}\mathfrak{f}_j^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-w_{\mathfrak{f}_i}^{sq}w_{\mathfrak{f}_j}^{tq}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}} \end{array}\right) \end{array}\right)$$

$$=\left(\begin{array}{l} S_{\left(\frac{1}{t}\left(\prod_{j=1}^{n}\left(t\theta_j\right)^{i}\right)^{\frac{2}{n(n+1)}}\right)}, \\ \left(\begin{array}{l} \left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-\mathfrak{t}_j^q\right)^t\right)^i\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\left(\prod_{i=1}^{n}\left(1-\left(1-w_{\mathfrak{t}_j}^q\right)^t\right)^i\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{t}}\right)^{\frac{1}{q}}}, \\ \left(1-\left(\prod_{i=1}^{n}\left(1-\mathfrak{f}_j^{tq}\right)^i\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(t)}} e^{i2\pi\left(1-\left(\prod_{i=1}^{n}\left(1-w_{\mathfrak{f}_j}^{tq}\right)^i\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(t)}}} \end{array}\right) \end{array}\right)$$

is called q-rung orthopair linguistic generalized geometric linear ascending weighted mean (q-ROLGGLAWM) operator.

3. When $s = t = 1$, then

$$Cq-ROLGHM^{1,1}(P_1,P_2,..,P_n)=$$

$$\left(\begin{array}{l} S_{\left(\frac{1}{s+t}\prod_{i=1}^{n}\prod_{j=1}^{n}\left(s\theta_i+t\theta_j\right)^{\frac{2}{n(n+1)}}\right)}, \\ \left(\begin{array}{l} \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{t}_i^q\right)^1\left(1-\mathfrak{t}_j^q\right)^1\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{2}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-w_{\mathfrak{t}_i}^q\right)^1\left(1-w_{\mathfrak{t}_j}^q\right)^1\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{2}}\right)^{\frac{1}{q}}}, \\ \left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathfrak{f}_i^{q}\mathfrak{f}_j^{q}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{2q}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-w_{\mathfrak{f}_i}^{q}w_{\mathfrak{f}_j}^{q}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{2q}}} \end{array}\right) \end{array}\right)$$

is called q-rung orthopair linguistic geometric line Heronian mean (q-ROLGLHM) operator.

4. When $q = 2$, then

$$Cq-ROLGHM^{s,t}(P_1,P_2,..,P_n)=$$
$$\left(\mathcal{S}_{\left(\frac{1}{s+t}\prod_{i=1}^{n}\prod_{j=1}^{n}\left(s\theta_i+t\theta_j\right)^{\frac{2}{n(n+1)}}\right)},\left(\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{t}_i^2\right)^s\left(1-\mathfrak{t}_j^2\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{2}}e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{t}_i}^2\right)^s\left(1-\mathcal{W}_{\mathfrak{t}_j}^2\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{2}}},\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathfrak{f}_i^{2s}\mathfrak{f}_j^{2t}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{2(s+t)}}e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathcal{W}_{\mathfrak{f}_i}^{2s}\mathcal{W}_{\mathfrak{f}_j}^{2t}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{2(s+t)}}}\right)\right)$$

is called pythagorean linguistic geometric Heromian mean (PLGHM) operator.

5. When $q = 1$, then

$$Cq-ROLGHM^{s,t}(P_1,P_2,..,P_n)=$$
$$\left(\mathcal{S}_{\left(\frac{1}{s+t}\prod_{i=1}^{n}\prod_{j=1}^{n}\left(s\theta_i+t\theta_j\right)^{\frac{2}{n(n+1)}}\right)},\left(\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{t}_i^1\right)^s\left(1-\mathfrak{t}_j^1\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{1}e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{t}_i}^1\right)^s\left(1-\mathcal{W}_{\mathfrak{t}_j}^1\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{1}},\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathfrak{f}_i^{s}\mathfrak{f}_j^{t}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{(s+t)}}e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\mathcal{W}_{\mathfrak{f}_i}^{s}\mathcal{W}_{\mathfrak{f}_j}^{t}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{(s+t)}}}\right)\right)$$

is called intuitionistic linguistic geometric Heronian mean (ILGHM) operator.

4.6. Complex q-Rung Orthopair Linguistic Weighted Geometric Heronian Mean (Cq-Rolwghm) Operators

Last sub-section, we proposed the Cq-ROLGHM operator without weight vectors. Therefore, we will propose the weighted form of the Cq-ROLGHM called Cq-ROLWGHM operator. Further, $\omega=(\omega_1,\omega_2,..,\omega_n)^T$ represented the weight vectors with $\sum_{i=1}^{n}\omega_i=1$.

Definition 18. Let $P_i=\left(\mathcal{S}_{\theta_i},\left(\mathfrak{t}_ie^{i2\pi\mathcal{W}_{\mathfrak{t}_i}},\mathfrak{f}_ie^{i2\pi\mathcal{W}_{\mathfrak{f}_i}}\right)\right),(i=1,2,\ldots,n)$ be a family of Cq-ROLNs, then the $Cq-ROLWGHM^{s,t}$ is defined: $Cq-ROLWGHM^{s,t}:\xi^n\to\xi$ by

$$Cq-ROLWGHM^{s,t}(P_1,P_2,..,P_n)=\left(\frac{1}{s+t}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(sP_i^{n\omega_i}+tP_j^{n\omega_j}\right)^{\frac{2}{n(n+1)}}\right) \tag{30}$$

where ξ^n is denoted the family of all Cq-ROLNs.

According to the operational laws of Cq-ROLNs, we can get the following result.

Theorem 13. *Let* $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), (i = 1, 2, \ldots, n)$ *be a family of Cq-ROLNs. Then we have*

$$Cq-ROLWGHM^{s,t}(P_1, P_2, .., P_n) = \left(\begin{array}{l} \mathcal{S}_{\left(\frac{1}{s+t}\prod_{i=1}^{n}\prod_{j=1}^{n}\left(s\theta_i^{n\omega_i}+t\theta_j^{n\omega_j}\right)^{\frac{2}{n(n+1)}}\right)}, \\ \left(\begin{array}{l} \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{t}_i^{n\omega_i q}\right)^s\left(1-\mathfrak{t}_j^{n\omega_j q}\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{t}_i}^{n\omega_i q}\right)^s\left(1-\mathcal{W}_{\mathfrak{t}_j}^{n\omega_j q}\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}}, \\ \left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\left(1-\mathfrak{f}_i^{q}\right)^{n\omega_i}\right)^s\left(1-\left(1-\mathfrak{f}_j^{q}\right)^{n\omega_j}\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^{q}\right)^{n\omega_i}\right)^s\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_j}^{q}\right)^{n\omega_j}\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}} \end{array}\right) \end{array}\right) \tag{31}$$

Proof: Similar to Theorem 1.

Example 5

We consider the four Cq-ROLNs $P_1 = \left(\mathcal{S}_{4.6}, \left(\begin{array}{l}0.97e^{i2\pi(0.97)},\\ 0.22e^{i2\pi(0.22)}\end{array}\right)\right)$, $P_2 = \left(\mathcal{S}_{2.3}, \left(\begin{array}{l}0.02e^{i2\pi(0.01)},\\ 0.12e^{i2\pi(0.25)}\end{array}\right)\right)$, $P_3 = \left(\mathcal{S}_{2.05}, \left(\begin{array}{l}0.95e^{i2\pi(0.98)},\\ 0.20e^{i2\pi(0.13)}\end{array}\right)\right)$ and $P_4 = \left(\mathcal{S}_{2.00}, \left(\begin{array}{l}0.96e^{i2\pi(0.95)},\\ 0.21e^{i2\pi(0.20)}\end{array}\right)\right)$, and suppose the parameters $q = 5, s = t = 1,\ n = 4$, then we have

$$A = Cq-ROLWGHM^{s,t}(P_1, P_2, .., P_n) = \left(\begin{array}{l} \mathcal{S}_{\left(\frac{1}{s+t}\prod_{i=1}^{n}\prod_{j=1}^{n}\left(s\theta_i^{n\omega_i}+t\theta_j^{n\omega_j}\right)^{\frac{2}{n(n+1)}}\right)}, \\ \left(\begin{array}{l} \left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathfrak{t}_i^{n\omega_i q}\right)^s\left(1-\mathfrak{t}_j^{n\omega_j q}\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}} e^{i2\pi\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\mathcal{W}_{\mathfrak{t}_i}^{n\omega_i q}\right)^s\left(1-\mathcal{W}_{\mathfrak{t}_j}^{n\omega_j q}\right)^t\right)^{\frac{2q}{n(n+1)}}\right)^{\frac{1}{s+t}}\right)^{\frac{1}{q}}}, \\ \left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\left(1-\mathfrak{f}_i^{q}\right)^{n\omega_i}\right)^s\left(1-\left(1-\mathfrak{f}_j^{q}\right)^{n\omega_j}\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}} e^{i2\pi\left(1-\prod_{i=1}^{n}\prod_{j=1}^{n}\left(1-\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_i}^{q}\right)^{n\omega_i}\right)^s\left(1-\left(1-\mathcal{W}_{\mathfrak{f}_j}^{q}\right)^{n\omega_j}\right)^t\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(s+t)}}} \end{array}\right) \end{array}\right)$$

$$= \left(\mathcal{S}_{2.415}, \left(\begin{array}{l}0.99e^{i2\pi(0.99)},\\ 0.009e^{i2\pi(0.008)}\end{array}\right)\right)$$

When we use the definition of score function, we get

$S(A) = 2.39$

Further, we discuss some properties of Cq-ROLNs as follows:

Theorem 14. *(Monotonicity) Let* $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right)$ *and* $Q_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), (i = 1, 2, \dots, n)$ *be two families of Cq-ROLNs, if* $P_i \leq Q_i$ *for all* $i = 1, 2, \dots, n$. *Then*

$$Cq - ROLWGHM^{s,t}(P_1, P_2, .., P_n) \leq Cq - ROLWGHM^{s,t}(Q_1, Q_2, .., Q_n)$$

Proof: Similar to Theorem 2.

Theorem 15. *(Idempotency) Let* $P_i = \left(\mathcal{S}_{\theta_i}, \left(\mathfrak{t}_i e^{i2\pi \mathcal{W}_{\mathfrak{t}_i}}, \mathfrak{f}_i e^{i2\pi \mathcal{W}_{\mathfrak{f}_i}}\right)\right), (i = 1, 2, \dots, n)$ *be a family of Cq-ROLNs, if* $P_i = P$ *for all* $i = 1, 2, \dots, n$. *Then*

$$Cq - ROLWGHM^{s,t}(P_1, P_2, .., P_n) = P$$

Proof: Similar to Theorem 3.

Theorem 16. *(Boundedness) The Cq-ROLWGHM operator lies between the max and min operators*

$$min(P_1, P_2, .., P_n) \leq Cq - ROLWGHM^{s,t}(P_1, P_2, .., P_n) \leq max(P_1, P_2, .., P_n)$$

Proof: Similar to Theorem 4.

5. A NEW MULTI-ATTRIBUTE GROUP DECISION-MAKING (MAGDM) METHOD

In this section, we would propose a new decision-making method with complex q-rung orthopair linguistic information. Consider the set of alternatives and the set of attributes with respect to weight vectors, i.e., $X = \{x_1, x_2, .., x_m\}$ isset of alternatives, $Y = \{y_1, y_2, .., y_n\}$ is set of attributes, and $\omega = (\omega_1, \omega_2, .., \omega_n)^T$ is the weight vector of the attributes such that $\sum_{i=1}^{n} \omega_i = 1$. Suppose decision makers are $D = \{D_1, D_2, .., D_p\}$, and decision maker D_k gives the evaluation value of attribute $y_j$for alternative x_{2i} which is expressed by Cq-ROLN is $P_{ij}^k = \left(\mathcal{S}_{\theta_{ij}}^k, \left(\mathfrak{t}_{ij}^k e^{i2\pi \mathcal{W}_{\mathfrak{t}_{ij}}^k}, \mathfrak{f}_{ij}^k e^{i2\pi \mathcal{W}_{\mathfrak{f}_{ij}}^k}\right)\right)$, and the complex q-rung orthopair linguistic decision matrices is represented by $A^k = \left(P_{ij}^k\right)_{m \times n}$. Then we will use two different operators to solve this problem. The procedure of the MAGDM is shown as follows:

1. Construct the decision matrices, it is necessary to consider two kinds of attribute like cost and benefits. The decision matrices is obtained by

$$P_{ij}^k = \begin{cases} \left(\mathcal{S}_{\theta_{ij}}^k, \left(\mathfrak{t}_{ij}^k e^{i2\pi \mathcal{W}_{\mathfrak{t}_{ij}}^k}, \mathfrak{f}_{ij}^k e^{i2\pi \mathcal{W}_{\mathfrak{f}_{ij}}^k}\right)\right) & y_j \in I_1 \\ \left(\mathcal{S}_{\theta_{ij}}^k, \left(\mathfrak{f}_{ij}^k e^{i2\pi \mathcal{W}_{\mathfrak{f}_{ij}}^k}, \mathfrak{t}_{ij}^k e^{i2\pi \mathcal{W}_{\mathfrak{t}_{ij}}^k}\right)\right) & y_j \in I_2 \end{cases} \tag{32}$$

   The symbol I_1 and I_2 represent the benefits and cost attributes.

2. Use the Cq-ROLWHM operator

$$P_{ij} = Cq - ROLWHM^{s,t}\left(P_{ij}^1, P_{ij}^2, .., P_{ij}^p\right)$$

   Or the Cq-ROLWHM operator

$$P_{ij} = Cq - ROLWGHM^{s,t}\left(P_{ij}^1, P_{ij}^2, .., P_{ij}^p\right)$$

   To aggregate the decision matrices $A^k = \left(P_{ij}^k\right)_{m \times n}$ into a single matrix $A = \left(P_{ij}\right)_{m \times n}$.

3. Use the Cq-ROLWHM operator

$$P_i = Cq - ROLWHM^{s,t}\left(P_{i1}, P_{i2}, .., P_{in}\right)$$

Or the Cq-ROLWHM operator

$$P_i = Cq - ROLWGHM^{s,t}\left(P_{i1}, P_{i2}, .., P_{in}\right)$$

To aggregate the decision matrices $A^k = \left(P_{ij}^k\right)_{m\times n}$ into a single value Cq-ROLN.

4. Calculate the score function and accuracy function of Cq-ROLNs.
5. Rank to all Cq-ROLNs and choose the best alternative.
6. End

Example 6

In this sub-section, we adopted a numerical example from [29] to show the application of the proposed method. The saving enterprise wants to invest its share with another enterprise. After search, there are four possible enterprises in the list of applicants which are

1. A_1: Car enterprise.
2. A_2: Computer enterprise.
3. A_3: TV enterprise.
4. A_4: Food enterprise.

The decision experts (D_1, D_2, and D_3) are invited to examine the candidates with respect attributes which are

1. C_1: Risk analysis.
2. C_2: Growth analysis.
3. C_3: Social–political impact analysis.
4. C_4: Environmental impact analysis.

Suppose the weight vector for attributes is $\omega = \{0.34, 0.32, 0.11, 0.23\}^T$ and the weight vector for decision experts is $\mho = \{0.45, 0.35, 0.20\}^T$. The decision experts adopt linguistic term set: $S = \{S_0 = \textit{very poor}, S_1 = \textit{poor}, S_2 = \textit{slightly poor}, S_3 = \textit{fair}, S_4 = \textit{slightly good}, S_5 = \textit{good}, S_6 = \textit{very good}\}$ to give the evaluation information shown in Tables 1–3.

5.1. Decision-Making Process

The steps of this decision-making problem are given as

1. The four attributes are all benefits types, so we cannot normalize the decision matrix.

Table 1 Complex q-rung orthopair linguistic decision matrix R^1 *by* D_1

Data Analysis	C_1	C_2	C_3	C_4
A_1	$\left(s_5, \left(0.2e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.3)}\right)\right)$	$\left(s_5, \left(0.5e^{i2\pi(0.7)}, 0.45e^{i2\pi(0.3)}\right)\right)$	$\left(s_5, \left(0.7e^{i2\pi(0.5)}, 0.2e^{i2\pi(0.4)}\right)\right)$	$\left(s_6, \left(0.4e^{i2\pi(0.7)}, 0.45e^{i2\pi(0.2)}\right)\right)$
A_2	$\left(s_3, \left(0.3e^{i2\pi(0.5)}, 0.7e^{i2\pi(0.4)}\right)\right)$	$\left(s_6, \left(0.55e^{i2\pi(0.45)}, 0.35e^{i2\pi(0.55)}\right)\right)$	$\left(s_4, \left(0.34e^{i2\pi(0.54)}, 0.6e^{i2\pi(0.4)}\right)\right)$	$\left(s_5, \left(0.3e^{i2\pi(0.45)}, 0.66e^{i2\pi(0.5)}\right)\right)$
A_3	$\left(s_3, \left(0.1e^{i2\pi(0.2)}, 0.8e^{i2\pi(0.6)}\right)\right)$	$\left(s_1, \left(0.45e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.3)}\right)\right)$	$\left(s_6, \left(0.22e^{i2\pi(0.53)}, 0.56e^{i2\pi(0.45)}\right)\right)$	$\left(s_4, \left(0.1e^{i2\pi(0.6)}, 0.9e^{i2\pi(0.3)}\right)\right)$
A_4	$\left(s_2, \left(0.4e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.3)}\right)\right)$	$\left(s_4, \left(0.34e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.4)}\right)\right)$	$\left(s_1, \left(0.1e^{i2\pi(0.56)}, 0.77e^{i2\pi(0.34)}\right)\right)$	$\left(s_3, \left(0.4e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.3)}\right)\right)$

2. We will consider the Cq-ROLWHM operator

$$P_{ij} = Cq - ROLWHM^{s,t}\left(P_{ij}^{1}, P_{ij}^{2}, .., P_{ij}^{p}\right)$$

To aggregate the decision matrices $A^k = \left(P_{ij}^k\right)_{m\times n}$ into a single matrix $A = \left(P_{ij}\right)_{m\times n}$ which is shown in Table 4 for $q = 3$.

3. We use the Cq-ROLWGHM operator

$$P_i = Cq - ROLWGHM^{s,t}\left(P_{i1}, P_{i2}, .., P_{in}\right)$$

To aggregate the decision matrix (in Table 4) and get the comprehensive value of four alternatives which is listed in Table 5.

4. Calculate the score functions of four alternatives which is listed in Table 6.
5. Rank all Cq-ROLNs and choose the best alternative.

$$A_1 \geq A_2 \geq A_4 \geq A_3$$

So, A_1 is the best alternative.

6. End.

Table 2 Complex q-rung orthopair linguistic decision matrix $R^2 by\, D_2$

Data Analysis	C_1	C_2	C_3	C_4
A_1	$\left(s_4, \left(\begin{matrix}0.1e^{i2\pi(0.7)},\\ 0.1e^{i2\pi(0.2)}\end{matrix}\right)\right)$	$\left(s_6, \left(\begin{matrix}0.55e^{i2\pi(0.71)},\\ 0.44e^{i2\pi(0.29)}\end{matrix}\right)\right)$	$\left(s_6, \left(\begin{matrix}0.72e^{i2\pi(0.52)},\\ 0.18e^{i2\pi(0.4)}\end{matrix}\right)\right)$	$\left(s_2, \left(\begin{matrix}0.3e^{i2\pi(0.4)},\\ 0.7e^{i2\pi(0.2)}\end{matrix}\right)\right)$
A_2	$\left(s_6, \left(\begin{matrix}0.55e^{i2\pi(0.45)},\\ 0.42e^{i2\pi(0.5)}\end{matrix}\right)\right)$	$\left(s_4, \left(\begin{matrix}0.53e^{i2\pi(0.46)},\\ 0.34e^{i2\pi(0.54)}\end{matrix}\right)\right)$	$\left(s_5, \left(\begin{matrix}0.46e^{i2\pi(0.57)},\\ 0.34e^{i2\pi(0.44)}\end{matrix}\right)\right)$	$\left(s_4, \left(\begin{matrix}0.5e^{i2\pi(0.3)},\\ 0.45e^{i2\pi(0.5)}\end{matrix}\right)\right)$
A_3	$\left(s_4, \left(\begin{matrix}0.45e^{i2\pi(0.6)},\\ 0.34e^{i2\pi(0.4)}\end{matrix}\right)\right)$	$\left(s_1, \left(\begin{matrix}0.46e^{i2\pi(0.62)},\\ 0.49e^{i2\pi(0.29)}\end{matrix}\right)\right)$	$\left(s_4, \left(\begin{matrix}0.44e^{i2\pi(0.54)},\\ 0.54e^{i2\pi(0.45)}\end{matrix}\right)\right)$	$\left(s_4, \left(\begin{matrix}0.2e^{i2\pi(0.1)},\\ 0.6e^{i2\pi(0.3)}\end{matrix}\right)\right)$
A_4	$\left(s_5, \left(\begin{matrix}0.34e^{i2\pi(0.6)},\\ 0.22e^{i2\pi(0.3)}\end{matrix}\right)\right)$	$\left(s_3, \left(\begin{matrix}0.35e^{i2\pi(0.61)},\\ 0.49e^{i2\pi(0.39)}\end{matrix}\right)\right)$	$\left(s_2, \left(\begin{matrix}0.12e^{i2\pi(0.58)},\\ 0.74e^{i2\pi(0.13)}\end{matrix}\right)\right)$	$\left(s_6, \left(\begin{matrix}0.3e^{i2\pi(0.4)},\\ 0.6e^{i2\pi(0.3)}\end{matrix}\right)\right)$

Table 3 Complex q-rung orthopair linguistic decision matrix R^3 $by\, D_3$

Data Analysis	C_1	C_2	C_3	C_4
A_1	$\left(s_5, \left(\begin{matrix}0.2e^{i2\pi(0.7)},\\ 0.45e^{i2\pi(0.3)}\end{matrix}\right)\right)$	$\left(s_5, \left(\begin{matrix}0.3e^{i2\pi(0.4)},\\ 0.7e^{i2\pi(0.2)}\end{matrix}\right)\right)$	$\left(s_6, \left(\begin{matrix}0.7e^{i2\pi(0.5)},\\ 0.2e^{i2\pi(0.4)}\end{matrix}\right)\right)$	$\left(s_5, \left(\begin{matrix}0.72e^{i2\pi(0.52)},\\ 0.18e^{i2\pi(0.4)}\end{matrix}\right)\right)$
A_2	$\left(s_6, \left(\begin{matrix}0.55e^{i2\pi(0.45)},\\ 0.35e^{i2\pi(0.55)}\end{matrix}\right)\right)$	$\left(s_6, \left(\begin{matrix}0.5e^{i2\pi(0.3)},\\ 0.45e^{i2\pi(0.5)}\end{matrix}\right)\right)$	$\left(s_4, \left(\begin{matrix}0.34e^{i2\pi(0.54)},\\ 0.6e^{i2\pi(0.5)}\end{matrix}\right)\right)$	$\left(s_3, \left(\begin{matrix}0.46e^{i2\pi(0.57)},\\ 0.34e^{i2\pi(0.44)}\end{matrix}\right)\right)$
A_3	$\left(s_4, \left(\begin{matrix}0.45e^{i2\pi(0.6)},\\ 0.5e^{i2\pi(0.3)}\end{matrix}\right)\right)$	$\left(s_1, \left(\begin{matrix}0.2e^{i2\pi(0.1)},\\ 0.6e^{i2\pi(0.3)}\end{matrix}\right)\right)$	$\left(s_4, \left(\begin{matrix}0.22e^{i2\pi(0.53)},\\ 0.56e^{i2\pi(0.45)}\end{matrix}\right)\right)$	$\left(s_6, \left(\begin{matrix}0.44e^{i2\pi(0.54)},\\ 0.54e^{i2\pi(0.45)}\end{matrix}\right)\right)$
A_4	$\left(s_6, \left(\begin{matrix}0.34e^{i2\pi(0.6)},\\ 0.5e^{i2\pi(0.4)}\end{matrix}\right)\right)$	$\left(s_4, \left(\begin{matrix}0.3e^{i2\pi(0.4)},\\ 0.6e^{i2\pi(0.3)}\end{matrix}\right)\right)$	$\left(s_1, \left(\begin{matrix}0.1e^{i2\pi(0.56)},\\ 0.77e^{i2\pi(0.34)}\end{matrix}\right)\right)$	$\left(s_1, \left(\begin{matrix}0.12e^{i2\pi(0.58)},\\ 0.74e^{i2\pi(0.13)}\end{matrix}\right)\right)$

Table 4 Complex q-rung orthopair linguistic decision matrix after using the Cq-ROLWHM operator

Data Analysis	C_1	C_2	C_3	C_4
A_1	$\left(s_{4.6}, \left(\begin{matrix}0.86e^{i2\pi(0.99)},\\ 0.10e^{i2\pi(0.03)}\end{matrix}\right)\right)$	$\left(s_{2.3}, \left(\begin{matrix}0.01e^{i2\pi(0.03)},\\ 0.25e^{i2\pi(0.03)}\end{matrix}\right)\right)$	$\left(s_{2.4}, \left(\begin{matrix}0.99e^{i2\pi(0.99)},\\ 0.01e^{i2\pi(0.09)}\end{matrix}\right)\right)$	$\left(s_{2.04}, \left(\begin{matrix}0.97e^{i2\pi(0.97)},\\ 0.18e^{i2\pi(0.04)}\end{matrix}\right)\right)$
A_2	$\left(s_{4.6}, \left(\begin{matrix}0.97e^{i2\pi(0.97)},\\ 0.22e^{i2\pi(0.22)}\end{matrix}\right)\right)$	$\left(s_{2.3}, \left(\begin{matrix}0.02e^{i2\pi(0.01)},\\ 0.12e^{i2\pi(0.25)}\end{matrix}\right)\right)$	$\left(s_{2.05}, \left(\begin{matrix}0.95e^{i2\pi(0.98)},\\ 0.20e^{i2\pi(0.13)}\end{matrix}\right)\right)$	$\left(s_{2.00}, \left(\begin{matrix}0.96e^{i2\pi(0.95)},\\ 0.21e^{i2\pi(0.20)}\end{matrix}\right)\right)$
A_3	$\left(s_{3.50}, \left(\begin{matrix}0.93e^{i2\pi(0.97)},\\ 0.34e^{i2\pi(0.15)}\end{matrix}\right)\right)$	$\left(s_{0.99}, \left(\begin{matrix}0.007e^{i2\pi(0.006)},\\ 0.25e^{i2\pi(0.06)}\end{matrix}\right)\right)$	$\left(s_{2,15}, \left(\begin{matrix}0.92e^{i2\pi(0.98)},\\ 0.24e^{i2\pi(0.14)}\end{matrix}\right)\right)$	$\left(s_{2.08}, \left(\begin{matrix}0.89e^{i2\pi(0.94)},\\ 0.53e^{i2\pi(0.08)}\end{matrix}\right)\right)$
A_4	$\left(s_{3.8}, \left(\begin{matrix}0.95e^{i2\pi(0.99)},\\ 0.15e^{i2\pi(0.09)}\end{matrix}\right)\right)$	$\left(s_{1.9}, \left(\begin{matrix}0.005e^{i2\pi(0.024)},\\ 0.25e^{i2\pi(0.09)}\end{matrix}\right)\right)$	$\left(s_{1.13}, \left(\begin{matrix}0.81e^{i2\pi(0.99)},\\ 0.54e^{i2\pi(0.033)}\end{matrix}\right)\right)$	$\left(s_{1.8}, \left(\begin{matrix}0.92e^{i2\pi(0.97)},\\ 0.37e^{i2\pi(0.04)}\end{matrix}\right)\right)$

Table 5 The comprehensive value of four alternatives

Data Analysis	$Cq-ROLN_s$
A_1	$\left(s_{3.06},\left(\begin{matrix}0.99e^{i2\pi(0.99)},\\ 0.021e^{i2\pi(0.09)}\end{matrix}\right)\right)$
A_2	$\left(s_{2.95},\left(\begin{matrix}0.99e^{i2\pi(0.99)},\\ 0.06e^{i2\pi(0.06)}\end{matrix}\right)\right)$
A_3	$\left(s_{2.21},\left(\begin{matrix}0.99e^{i2\pi(0.99)},\\ 0.18e^{i2\pi(0.02)}\end{matrix}\right)\right)$
A_4	$\left(s_{2.4},\left(\begin{matrix}0.99e^{i2\pi(0.99)},\\ 0.16e^{i2\pi(0.006)}\end{matrix}\right)\right)$

Table 6 The score function for four alternatives

$Cq-ROLN_s$	Score Function	Ranking
A_1	$S(A_1)=3.02$	*first*
A_2	$S(A_2)=2.76$	*second*
A_3	$S(A_3)=1.99$	*fourth*
A_4	$S(A_4)=2.16$	*thrid*

Table 7 Validation test

Methods	Score Function	Ranking
CIFPA operator proposed by Rani and Garg [35]	$S(A_1)=0.75, S(A_2)=0.73,$ $S(A_3)=0.64, S(A_4)=0.67$	$A_1>A_2>A_4>A_3$
CIFWA operator proposed by Garg and Rani [33]	$S(A_1)=1.99, S(A_2)=1.98,$ $S(A_3)=1.94, S(A_4)=1.95$	$A_1>A_2>A_4>A_3$
WDM for PYFS proposed by Ullah et al. [36]	$S(A_1)=0.56, S(A_2)=0.093,$ $S(A_3)=0.089, S(A_4)=0.09$	$A_1>A_2>A_4>A_3$
Method based on Cq-ROLS in this paper for $\boldsymbol{q=1}$	$S(A_1)=4.86, S(A_2)=3.6,$ $S(A_3)=2.19, S(A_4)=2.57$	$A_1>A_2>A_4>A_3$
Method based on Cq-ROLS in this paper for $\boldsymbol{q=2}$	$S(A_1)=3.6, S(A_2)=3.04,$ $S(A_3)=2.05, S(A_4)=2.3$	$A_1>A_2>A_4>A_3$
Method based on Cq-ROLS in this paper	$S(A_1)=2.47, S(A_2)=2.39,$ $S(A_3)=1.88, S(A_4)=1.99$	$A_1>A_2>A_4>A_3$

In order to explain the validity of the proposed method, we use the method for CILS and the method for CPYLS, the ranking results are listed in Table 7.

The geometrical interpretation of the proposed method with existing methods are discussed in Figure 3.

From Table 7, we can get the same ranking result, it can explain the validity of the proposed method.

6. ADVANTAGES AND COMPARATIVE ANALYSIS

6.1. The Influence of Parameters on Ranking Results

The parameters in the proposed operators play a key role on the final ranking results. By example 6, we assign different values to parameters s and t, and discuss the ranking results which are shown in Table 8.

From Table 8, we can see that although the best choice is the same, the ranking order is different, this can explain the parameters s and t can affect the ranking results.

In order to show clearly the ranking results, we consider the values of parameters for $s = t$, then the score values of alternatives $A_i\,(i=1,2,3,4)$ are shown in Figure 4.

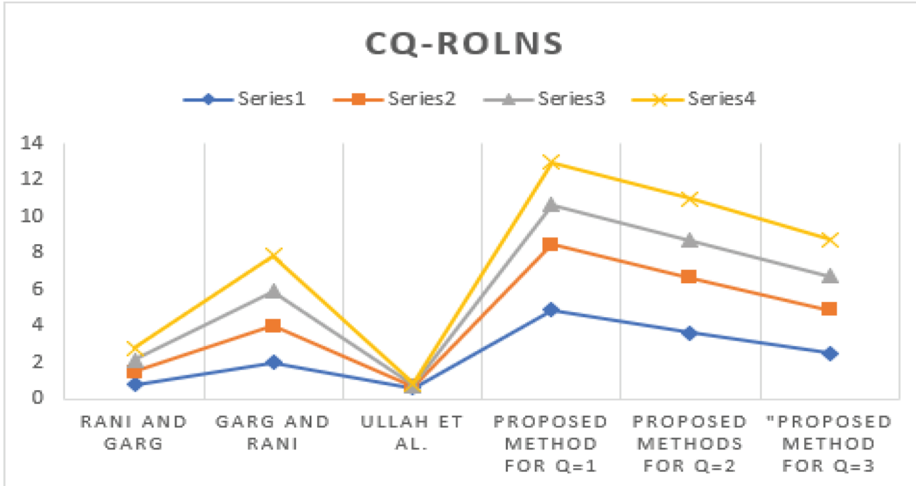

Figure 3 Geometrical interpretation for proposed and existing methods.

Table 8 Ranking results for different values of parameters

Parameters Value	Using Score Function	Ranking	Best Alternative
$s \to 0, t = 1$	$S(A_1) = 30.85, S(A_2) = 27.53,$ $S(A_3) = 11.54, S(A_4) = 14.00$	$A_1 > A_2 > A_4 > A_3$	A_1
$s = 1, t \to 0$	$S(A_1) = 21.97, S(A_2) = 20.43,$ $S(A_3) = 9.49, S(A_4) = 11.68$	$A_1 > A_2 > A_4 > A_3$	A_1
$s = 1, t = 1$	$S(A_1) = 3.02, S(A_2) = 2.76,$ $S(A_3) = 1.99, S(A_4) = 2.16$	$A_1 > A_2 > A_4 > A_3$	A_1
$s = 1, t = 0.5$	$S(A_1) = 5.12, S(A_2) = 4.76,$ $S(A_3) = 3.07, S(A_4) = 3.45$	$A_1 > A_2 > A_4 > A_3$	A_1
$s = 1, t = 2$	$S(A_1) = 1.94, S(A_2) = 1.69,$ $S(A_3) = 1.33, S(A_4) = 1.42$	$A_1 > A_2 > A_4 > A_3$	A_1
$s = 2, t = 3$	$S(A_1) = 1.32, S(A_2) = 1.07,$ $S(A_3) = 0.93, S(A_4) = 1.03$	$A_1 > A_2 > A_4 > A_3$	A_1
$s = 3, t = 4$	$S(A_1) = 1.12, S(A_2) = 0.86,$ $S(A_3) = 0.79, S(A_4) = 0.90$	$A_1 > A_4 > A_2 > A_3$	A_1
$s = t = 5$	$S(A_1) = 0.98, S(A_2) = 0.72,$ $S(A_3) = 0.70, S(A_4) = 0.81$	$A_1 > A_4 > A_2 > A_3$	A_1
$s = t = 6$	$S(A_1) = 0.93, S(A_2) = 0.664,$ $S(A_3) = 0.660, S(A_4) = 0.78$	$A_1 > A_4 > A_2 > A_3$	A_1
$s = t = 7$	$S(A_1) = 0.89, S(A_2) = 0.628,$ $S(A_3) = 0.635, S(A_4) = 0.75$	$A_1 > A_4 > A_2 > A_3$	A_1

6.2. Advantages of the Proposed Cq-ROLS with the Existing CFSs

The HM operators for CILS and CPYLS are also the special cases of our proposed method. The following examples can explain the generalization of the proposed Cq-ROLS.

Example 7

In some practical examples, the CILS cannot described effectively, because the restriction of of CILS is that the sum of membership (for real part and imaginary part) and non-membership (for real part and imaginary part) are limited to 1. So we considered the complex Pythagorean linguistic kinds of information, and solved by our proposed methods and then compared with existing methods. The weight vectors are given by $\omega = \{0.34, 0.32, 0.11, 0.23\}^T$. The complex Pythagorean linguistic decision matrix R shown in Table 9.

The aggregation results for different approaches shown in Table 10.

From Table 10, we can get (1) CILS cannot express the information described by CPYLS; (2) the proposed method in this paper can the same ranking results as method in [33], which can show the effectiveness of the proposed method because the Cq-ROLS is reduced into CPYFS when $q = 2, s = t = 5$.

Example 8

In this example, we consider the information is expressed by Cq-ROLNs which is listed in Table 11, and the weight vectors is taken from example 6.

Then the ranking results are listed in Table 12 $\left(for\, q = 5, s = t = 1\right)$.

From Table 12, we can know the Cq-ROLS is more generalized than existing CFSs, so we easily find that our proposed method is more superior and more reliable than existing methods.

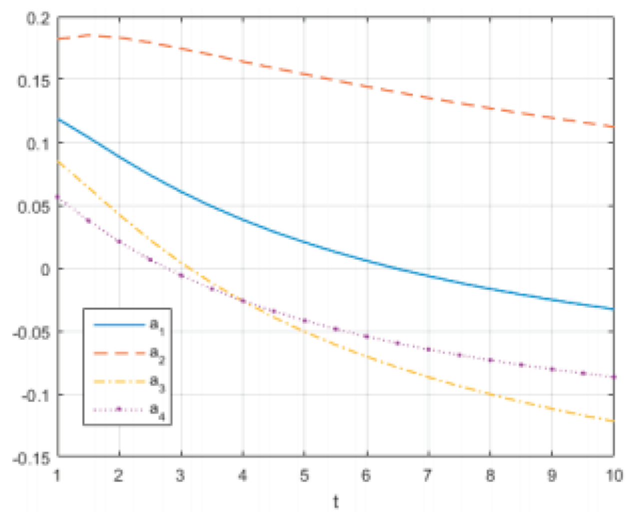

Figure 4 Scores of alternatives for parameters s and t

Table 9 Decision matrix for complex Pythagorean linguistic information's

Data Analysis	C_1	C_2	C_3	C_4
A_1	$\left(s_{1.34}, \left(\begin{matrix} 0.67e^{i2\pi(0.91)}, \\ 0.2e^{i2\pi(0.1)} \end{matrix}\right)\right)$	$\left(s_4, \left(\begin{matrix} 0.8e^{i2\pi(0.66)}, \\ 0.4e^{i2\pi(0.13)} \end{matrix}\right)\right)$	$\left(s_3, \left(\begin{matrix} 0.66e^{i2\pi(0.78)}, \\ 0.67e^{i2\pi(0.34)} \end{matrix}\right)\right)$	$\left(s_2, \left(\begin{matrix} 0.5e^{i2\pi(0.7)}, \\ 0.6e^{i2\pi(0.5)} \end{matrix}\right)\right)$
A_2	$\left(s_{1.33}, \left(\begin{matrix} 0.92e^{i2\pi(0.12)}, \\ 0.13e^{i2\pi(0.5)} \end{matrix}\right)\right)$	$\left(s_5, \left(\begin{matrix} 0.88e^{i2\pi(0.89)}, \\ 0.33e^{i2\pi(0.3)} \end{matrix}\right)\right)$	$\left(s_4, \left(\begin{matrix} 0.56e^{i2\pi(0.68)}, \\ 0.5e^{i2\pi(0.6)} \end{matrix}\right)\right)$	$\left(s_5, \left(\begin{matrix} 0.6e^{i2\pi(0.9)}, \\ 0.5e^{i2\pi(0.12)} \end{matrix}\right)\right)$
A_3	$\left(s_{1.27}, \left(\begin{matrix} 0.78e^{i2\pi(0.93)}, \\ 0.15e^{i2\pi(0.11)} \end{matrix}\right)\right)$	$\left(s_3, \left(\begin{matrix} 0.8e^{i2\pi(0.78)}, \\ 0.23e^{i2\pi(0.4)} \end{matrix}\right)\right)$	$\left(s_5, \left(\begin{matrix} 0.45e^{i2\pi(0.67)}, \\ 0.7e^{i2\pi(0.4)} \end{matrix}\right)\right)$	$\left(s_4, \left(\begin{matrix} 0.7e^{i2\pi(0.8)}, \\ 0.4e^{i2\pi(0.4)} \end{matrix}\right)\right)$
A_4	$\left(s_{1.29}, \left(\begin{matrix} 0.9e^{i2\pi(0.91)}, \\ 0.2e^{i2\pi(0.1)} \end{matrix}\right)\right)$	$\left(s_2, \left(\begin{matrix} 0.78e^{i2\pi(0.67)}, \\ 0.34e^{i2\pi(0.5)} \end{matrix}\right)\right)$	$\left(s_1, \left(\begin{matrix} 0.7e^{i2\pi(0.6)}, \\ 0.4e^{i2\pi(0.5)} \end{matrix}\right)\right)$	$\left(s_3, \left(\begin{matrix} 0.56e^{i2\pi(0.6)}, \\ 0.5e^{i2\pi(0.6)} \end{matrix}\right)\right)$

Table 10 Ranking results for proposed and existing methods to solve Example 7

Methods	Score Function	Ranking
CIFPA operator proposed by Rani and Garg [35]	*Cannot be calculated*	*Cannot be calculated*
CIFWA operator proposed by Garg and Rani [33]	*Cannot be calculated*	*Cannot be calculated*
WDM for CPYFS proposed by Ullah *et al.* [36]	$S\left(A_1\right) = 0.76, S\left(A_2\right) = 0.67,$ $S\left(A_3\right) = 0.45, S\left(A_4\right) = 0.54$	$A_1 > A_2 > A_4 > A_3$
Method based on Cq-ROLS in this paper for $q = 2$	$S\left(A_1\right) = 0.60, S\left(A_2\right) = 0.75,$ $S\left(A_3\right) = 0.66, S\left(A_4\right) = 0.50$	$A_2 > A_3 > A_1 > A_4$
Method based on Cq-ROLS in this paper	$S\left(A_1\right) = 0.8, S\left(A_2\right) = 1.01,$ $S\left(A_3\right) = 0.91, S\left(A_4\right) = 0.64$	$A_2 > A_3 > A_1 > A_4$

cq-ROLS, complex q-rung orthopair linguistic set.

Example 9

In this example, we consider the information expressed by Pythagorean linguistic sets, which is listed in Table 13, and the weight vectors is taken from example 6. The information discussed in this example is taken from [29].

We will convert the Table 13 into Table 14, and we also clear that about $e^0 = 1$.

Then the ranking results are listed in Table 15 $\left(for\, q = 5, s = t = 1\right)$.

From Table 15, we can know the Cq-ROLS is more generalized than existing CPYLS, CILS, q-ROLS, PYLS, ILS and CFSs, so we easily find that our proposed method is more superior and more reliable than existing methods. Therefore, the proposed method is more generalized than existing to cope with uncertain and complicated types of information easily.

6.3. The Qualitative Comparison with the Existing Methods

In this sub-section, we give some comparisons with some existing methods from a qualitative point of view. We compare our method with the work proposed by Ullah et al. [36] based on the similarity measures for complex PFS, the method proposed by Rani and Garg [35,37] based on the distance measures and power aggregation operators for CIFS, the method proposed by Garg and Rani [38,33] based on some

Table 11 | The decision matrix from Example 8

Data Analysis	C_1	C_2	C_3	C_4
A_1	$\left(s_{4.6}, \left(\begin{matrix}0.86e^{i2\pi(0.99)},\\ 0.10e^{i2\pi(0.03)}\end{matrix}\right)\right)$	$\left(s_{2.3}, \left(\begin{matrix}0.01e^{i2\pi(0.03)},\\ 0.25e^{i2\pi(0.03)}\end{matrix}\right)\right)$	$\left(s_{2.4}, \left(\begin{matrix}0.99e^{i2\pi(0.99)},\\ 0.01e^{i2\pi(0.09)}\end{matrix}\right)\right)$	$\left(s_{2.04}, \left(\begin{matrix}0.97e^{i2\pi(0.97)},\\ 0.18e^{i2\pi(0.04)}\end{matrix}\right)\right)$
A_2	$\left(s_{4.6}, \left(\begin{matrix}0.97e^{i2\pi(0.97)},\\ 0.22e^{i2\pi(0.22)}\end{matrix}\right)\right)$	$\left(s_{2.3}, \left(\begin{matrix}0.02e^{i2\pi(0.01)},\\ 0.12e^{i2\pi(0.25)}\end{matrix}\right)\right)$	$\left(s_{2.05}, \left(\begin{matrix}0.95e^{i2\pi(0.98)},\\ 0.20e^{i2\pi(0.13)}\end{matrix}\right)\right)$	$\left(s_{2.00}, \left(\begin{matrix}0.96e^{i2\pi(0.95)},\\ 0.21e^{i2\pi(0.20)}\end{matrix}\right)\right)$
A_3	$\left(s_{3.50}, \left(\begin{matrix}0.93e^{i2\pi(0.97)},\\ 0.34e^{i2\pi(0.15)}\end{matrix}\right)\right)$	$\left(s_{0.99}, \left(\begin{matrix}0.007e^{i2\pi(0.006)},\\ 0.25e^{i2\pi(0.06)}\end{matrix}\right)\right)$	$\left(s_{2,15}, \left(\begin{matrix}0.92e^{i2\pi(0.98)},\\ 0.24e^{i2\pi(0.14)}\end{matrix}\right)\right)$	$\left(s_{2.08}, \left(\begin{matrix}0.89e^{i2\pi(0.94)},\\ 0.53e^{i2\pi(0.08)}\end{matrix}\right)\right)$
A_4	$\left(s_{3.8}, \left(\begin{matrix}0.95e^{i2\pi(0.99)},\\ 0.15e^{i2\pi(0.09)}\end{matrix}\right)\right)$	$\left(s_{1.9}, \left(\begin{matrix}0.005e^{i2\pi(0.024)},\\ 0.25e^{i2\pi(0.09)}\end{matrix}\right)\right)$	$\left(s_{1.13}, \left(\begin{matrix}0.81e^{i2\pi(0.99)},\\ 0.54e^{i2\pi(0.033)}\end{matrix}\right)\right)$	$\left(s_{1.8}, \left(\begin{matrix}0.92e^{i2\pi(0.97)},\\ 0.37e^{i2\pi(0.04)}\end{matrix}\right)\right)$

Table 12 | Ranking results from different complex fuzzy sets for Example 8

Methods	Score Function	Ranking
Complex intuitionistic fuzzy power averaging (CIFPA) aggregation operator proposed by Rani and Garg [35]	*Cannot be calculated*	*Cannot be calculated*
Complex intuitionistic fuzzy weighted averaging (CIFWA) operator proposed by Garg and Rani [33]	*Cannot be calculated*	*Cannot be calculated*
Weighted distance measure (WDM) for complex pythagorean fuzzy set (CPYFS) proposed by Ullah *et al.* [36]	*Cannot be calculated*	*Cannot be calculated*
Method based on Cq-ROLS in this paper	$S\left(A_1\right) = 2.48, S\left(A_2\right) = 2.39,$ $S\left(A_3\right) = 1.88, S\left(A_4\right) = 1.99$	$A_1 > A_2 > A_4 > A_3$

cq-ROLS, complex q-rung orthopair linguistic set.

Table 13 | The decision matrix from Example 9

Data Analysis	C_1	C_2	C_3	C_4
A_1	$\left(s_{4.7038}, \left(\begin{matrix}0.1820,\\ 0.6711\end{matrix}\right)\right)$	$\left(s_{2.6020}, \left(\begin{matrix}0.3377,\\ 0.6650\end{matrix}\right)\right)$	$\left(s_{4.1372}, \left(\begin{matrix}0.4235,\\ 0.5997\end{matrix}\right)\right)$	$\left(s_{5.0751}, \left(\begin{matrix}0.3067,\\ 0.5992\end{matrix}\right)\right)$
A_2	$\left(s_{4.3333}, \left(\begin{matrix}0.3796,\\ 0.5992\end{matrix}\right)\right)$	$\left(s_{4.4066}, \left(\begin{matrix}0.3515,\\ 0.5672\end{matrix}\right)\right)$	$\left(s_{3.7082}, \left(\begin{matrix}0.1533,\\ 0.7358\end{matrix}\right)\right)$	$\left(s_{3.4366}, \left(\begin{matrix}0.4235,\\ 0.5999\end{matrix}\right)\right)$
A_3	$\left(s_{3.6013}, \left(\begin{matrix}0.2002,\\ 0.6686\end{matrix}\right)\right)$	$\left(s_{4.2846}, \left(\begin{matrix}0.2396,\\ 0.6711\end{matrix}\right)\right)$	$\left(s_{2.6550}, \left(\begin{matrix}0.3237,\\ 0.6979\end{matrix}\right)\right)$	$\left(s_{4.1372}, \left(\begin{matrix}0.2450,\\ 0.7011\end{matrix}\right)\right)$
A_4	$\left(s_{4.9334}, \left(\begin{matrix}0.4084,\\ 0.5621\end{matrix}\right)\right)$	$\left(s_{2.9382}, \left(\begin{matrix}0.3018,\\ 0.6743\end{matrix}\right)\right)$	$\left(s_{2.9832}, \left(\begin{matrix}0.2703,\\ 0.6020\end{matrix}\right)\right)$	$\left(s_{3.8820}, \left(\begin{matrix}0.3199,\\ 0.5710\end{matrix}\right)\right)$

Table 14 The decision matrix from Example 9

Data Analysis	C_1	C_2	C_3	C_4
A_1	$\left(s_{4.7038}, \left(\begin{array}{c}0.1820e^{i2\pi(0.0)},\\ 0.6711e^{i2\pi(0.0)}\end{array}\right)\right)$	$\left(s_{2.6020}, \left(\begin{array}{c}0.3377e^{i2\pi(0.0)},\\ 0.6650e^{i2\pi(0.0)}\end{array}\right)\right)$	$\left(s_{4.1372}, \left(\begin{array}{c}0.4235e^{i2\pi(0.0)},\\ 0.5997e^{i2\pi(0.0)}\end{array}\right)\right)$	$\left(s_{5.0751}, \left(\begin{array}{c}0.3067e^{i2\pi(0.0)},\\ 0.5992e^{i2\pi(0.0)}\end{array}\right)\right)$
A_2	$\left(s_{4.3333}, \left(\begin{array}{c}0.3796e^{i2\pi(0.0)},\\ 0.5992e^{i2\pi(0.0)}\end{array}\right)\right)$	$\left(s_{4.4066}, \left(\begin{array}{c}0.3515e^{i2\pi(0.0)},\\ 0.5672e^{i2\pi(0.0)}\end{array}\right)\right)$	$\left(s_{3.7082}, \left(\begin{array}{c}0.1533e^{i2\pi(0.0)},\\ 0.7358e^{i2\pi(0.0)}\end{array}\right)\right)$	$\left(s_{3.4366}, \left(\begin{array}{c}0.4235e^{i2\pi(0.0)},\\ 0.5999e^{i2\pi(0.0)}\end{array}\right)\right)$
A_3	$\left(s_{3.6013}, \left(\begin{array}{c}0.2002e^{i2\pi(0.0)},\\ 0.6686e^{i2\pi(0.0)}\end{array}\right)\right)$	$\left(s_{4.2846}, \left(\begin{array}{c}0.2396e^{i2\pi(0.0)},\\ 0.6711e^{i2\pi(0.0)}\end{array}\right)\right)$	$\left(s_{2.6550}, \left(\begin{array}{c}0.3237e^{i2\pi(0.0)},\\ 0.6979e^{i2\pi(0.0)}\end{array}\right)\right)$	$\left(s_{4.1372}, \left(\begin{array}{c}0.2450e^{i2\pi(0.0)},\\ 0.7011e^{i2\pi(0.0)}\end{array}\right)\right)$
A_4	$\left(s_{4.9334}, \left(\begin{array}{c}0.4084e^{i2\pi(0.0)},\\ 0.5621e^{i2\pi(0.0)}\end{array}\right)\right)$	$\left(s_{2.9382}, \left(\begin{array}{c}0.3018e^{i2\pi(0.0)},\\ 0.6743e^{i2\pi(0.0)}\end{array}\right)\right)$	$\left(s_{2.9832}, \left(\begin{array}{c}0.2703e^{i2\pi(0.0)},\\ 0.6020e^{i2\pi(0.0)}\end{array}\right)\right)$	$\left(s_{3.8820}, \left(\begin{array}{c}0.3199e^{i2\pi(0.0)},\\ 0.5710e^{i2\pi(0.0)}\end{array}\right)\right)$

Table 15 Ranking results from different complex fuzzy sets for Example 8

Methods	Score Function	Ranking
CIFPA operator proposed by Rani and Garg [35]	*Cannot be calculated*	*Cannot be calculated*
CIFWA operator proposed by Garg and Rani [33]	*Cannot be calculated*	*Cannot be calculated*
WDM for PYLS proposed in [29]	$S(A_1) = 0.9283, S(A_2) = 1.09,$ $S(A_3) = 0.9210, S(A_4) = 0.9176$	$A_2 > A_1 > A_4 > A_3$
Method based on Cq-ROLS in this paper	$S(A_1) = 0.8472, S(A_2) = 0.9982,$ $S(A_3) = 0.8356, S(A_4) = 0.8355$	$A_2 > A_1 > A_3 > A_4$

cq-ROLS, complex q-rung orthopair linguistic set.

Table 16 Comparison between existing methods and the proposed method

Methods	Ability to Integrate Information	Generalized Operators Based on t-norm and t-conorm	Ability to Capture Information Using Complex Numbers	Ability to Handle Two-dimensional Information	Flexible According to Decision maker's Preferences	Superior Characteristic of the Ideas
Zhang [39]	*No*	*Yes*	*No*	*No*	*Yes*	*No*
Liu [40]	*No*	*Yes*	*No*	*No*	*Yes*	*No*
Garg and Kumar [34]	*No*	*Yes*	*No*	*No*	*Yes*	*No*
Garg [41]	*No*	*Yes*	*No*	*No*	*Yes*	*No*
Garg and Rani [38,33]	*Yes*	*Yes*	*Yes*	*Yes*	*Yes*	*No*
Ullah *et al.* [36]	*Yes*	*Yes*	*Yes*	*Yes*	*Yes*	*No*
Rani and Garg [35,37]	*Yes*	*Yes*	*Yes*	*Yes*	*Yes*	*No*
Li *et al.* [29]	*No*	*Yes*	*No*	*No*	*Yes*	*No*
The proposed method for $q = 1$	*Yes*	*Yes*	*Yes*	*Yes*	*Yes*	*No*
The proposed method for $q = 2$	*Yes*	*Yes*	*Yes*	*Yes*	*Yes*	*No*
The proposed method for $q = 3$	*Yes*	*Yes*	*Yes*	*Yes*	*Yes*	*Yes*

robust correlation coefficient and generalized CIFS and their aggregation operators. The characteristic comparison of the proposed method with existing works is shown in Table 16.

From Table 16, it is clear that our proposed method is more superior than existing works because the CILS and CPLS are only special cases of Cq-ROLS, which is the generalization of ILS and PLS.

The idea of Cq-ROFLS is more powerful and more general than existing methods, from the above analysis we have clear, if we take the values of parameter $q = 1$, the proposed approach is converted to CIFLS and similarly, if we take the values of parameter $q = 2$, the proposed approach is converted to CPFLS. We discussed two numerical examples for existing methods and solved by proposed approach. The comparison between proposed methods and existing methods are discussed in Table 16, to show the reliability and effectiveness of the proposed methods. Hence, the introduced methods in this manuscript is more powerful and more general than existing methods.

7. CONCLUSION

The notions of Cq-ROFS and LV are two different tools to describe uncertain and unpredictable information in MAGDM problems. Motivation of this paper is to propose a new concept, called Cq-ROLS to cope with unreliable and difficult information in real decision problems, which takes full benefits of Cq-ROFS and LV. Futher, we generalize the HM operator to Cq-ROLS and propose Cq-ROLHM, Cq-ROLWHM, Cq-ROLGHM, Cq-ROLWGHM operators and discuss their properties in detail. Moreover, we develop a novel approach to MAGDM using proposed operators. We also use a numerical example to describe the flexibility and explicitly of the proposed method. In last, the comparisons between proposed method and existing methods are also discussed in detail.

In the future, we will use the proposed method to solve some real decision problems [42], such as Efficiency evaluation [43], ecological environment quality assessment [44], supplier selection problems [45], and so on. We can also extend the IFSs [46], interval-valued intuitionistic fuzzy sets [47], to their complex types.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

AUTHORS' CONTRIBUTIONS

Conceptualization, P.L., Z.A., T.M. (Peide Liu, Zeeshan Ali and Tahir Mahmood); methodology, P.L., Z.A., T.M.; software, Z.A.; validation, P.L., Z.A., T.M.; formal analysis, P.L., Z.A., T.M.; investigation, P.L., Z.A., T.M.; writing–original draft preparation, Z.A.; writing–review and editing, Z.A.; visualization, P.L., Z.A., T.M.; supervision, T.M.; funding acquisition, P.L.

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