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Integer Programming and Sports Rankings

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Abstract

Sports rankings are obtained by applying a system of rules to evaluate the performance of the participants in a competition. We consider rankings that result from assigning an ordinal rank to each competitor according to their performance. We develop an integer programming model for rankings that allows us to calculate the number of points needed to guarantee a team the i th position, as well as the minimum number of points that could yield the i th place. The model is very general and can thus be applied to many types of sports. We discuss examples coming from football (soccer), ice hockey, and Formula 1. We answer various questions and debunk a few myths along the way. Are 40 points enough to avoid relegation in the Bundesliga? Do 95 points guarantee the participation of a team in the NHL playoffs? Moreover, in the season restructuring currently under consideration in the NHL, will it be easier or harder to access the playoffs? Is it possible to win the Formula 1 World Championship without winning at least one race or without even climbing once on the podium? Finally, we observe that the optimal solutions of the aforementioned model are associated to extreme situations which are unlikely to happen. Thus, to get closer to realistic scenarios, we enhance the model by adding some constraints inferred from the results of the previous years.

1 Introduction

Sports fans are irrational creatures who swear eternal love to many idols and interpret the Bible of Statistics in whatever way they please. Their belief system is also filled with myths, some of which we will debunk by using a simple integer program.

For instance, it is widely believed among fans (and teams!) of the Bundesliga that a team that collects 40 points during a season is guaranteed to avoid relegation to the second league. Indeed, since the creation of the Bundesliga in 1963, no team

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that gathered at least 40 points (according to the modern point system which was introduced in 1995) was ever relegated. A similarly tenacious myth among fans of the NHL states that earning 95 points during the regular season guarantees a team a spot in the playoffs.

We will see that these beliefs, although well-established from previous experiences, do not hold strictly. Indeed, there have been counterexamples in minor leagues in Germany and even in the NHL itself, and it is quite easy to come up with theoretical seasons for the respective leagues in which each of these myths is thoroughly shattered. Still the question remains if there are such theoretical point bounds and, if so, what then really is the number of points necessary in the worst case. We will provide means to answer these questions using a general integer program (IP) and by offering ways to adapt it to different problems. This program also allows us to answer similar questions in other sports; we give an example of an application to Formula 1.

The most prominent sports problems are the traveling tournament problem and the referee assignment problem which were tackled by local search, integer programming, constraint programming, and tailor-made heuristic approaches. An annotated bibliography for sports scheduling over the past 40 years is provided by Kendall et al. [5]. Many articles ([2], [3], [8], [9]) also try to answer the question of whether or not, at some point in the season, some team has a chance to win (or move on) in a competition, and Gusfield & Martel [4] as well as Kern & Paulusma [6] discuss the complexity of this problem. For example, Russell & van Beek [9] use constraint programming to determine the number of games needed to guarantee a playoff spot in the NHL at any point in the season. They do mention very briefly that for the specific schedule of 2006/2007, the teams of Toronto and Pittsburgh need 145 points at the beginning of the season to ensure a playoff spot; however, they do not discuss the maximum number of points that might be needed in general for any schedule and any team. To the best of the authors' knowledge, there are no published articles about a general ranking integer programming model that calculates the number of points needed to finish i th in any sport.

In the following Section 2, we make an introductory example from the Bundesliga precise and straighten out the 40-point rule. We also introduce the general IP model. In Section 3, we adapt this model to the setting of Formula 1 and provide some surprising results about what is possible in a racing season. Applying the model to the NHL is much more intricate and we discuss this in Section 4; here we also consider the restructuring currently under discussion in the NHL. Incorporating “experience” constraints that reflect what usually happens during a season rules out the most extreme scenarios and gives bounds that might be closer to reality. Finally, Section 5 gives a short summary of the results. All computed results mentioned throughout the paper, as well as the model files, can be obtained from a website, see Section 5.

2 The Bundesliga and the General Model

Many German football (soccer) fans believe that if a team in the Bundesliga earns 40 points, then it will not be relegated to the second league. The Bundesliga has the following structure: It is composed of 18 teams which each play against each other team twice in the season, once at home and once away. A win earns a team three points, and a tie, one point. The two worst teams of the Bundesliga are relegated to the second league, whereas the top two teams of the second league are promoted to the Bundesliga. The third worst team of the Bundesliga and the third best team of the second league affront each other in an extra game; the winner plays its next season in the Bundesliga, and the loser, in the second league. Thus the myth among the fans says that 40 points guarantee a team at least the fifteenth place. We can easily check the validity of this statement by formulating an IP.

Let $T := \{1, \dots, 18\}$ be the set of teams and G be the set of games $g = (t, t', n)$, where n denotes whether g is the first or second game between teams $t, t' \in T$. Then we can formulate the following integer program:

$$\begin{aligned}
& \max && p_{16} \\
& \text{s.t.} && x_g^0 + x_g^1 + x_g^2 = 1 && \forall g \in G \\
& && \sum_{\substack{g \in G: \\ g=(t,t',n)}} (3x_g^1 + x_g^0) + \sum_{\substack{g \in G: \\ g=(t',t,n)}} (3x_g^2 + x_g^0) = p_t && \forall t \in T \\
& && p_{t+1} \leq p_t && \forall t \in T \setminus \{18\} \\
& && x_g^i \in \{0, 1\} && \forall g \in G, i \in \{0, 1, 2\}
\end{aligned}$$

Here, $x_g^i = 1$ for game $g = (t, t', n)$ if $i = 0$ and g is a tie, if $i = 1$ and t wins g or if $i = 2$ and t' wins g ; otherwise, $x_g^i = 0$. The first equation states that game $g = (t, t', n)$ ends either with the victory of t or t' or with a tie. Moreover, the second equation counts p_t , the number of points earned by team t . Note that, by the third inequality, the teams are labeled from 1 to 18 in order of their rank at the end of the season. Finally, by maximizing p_{16} , we find out the maximum number of points that a team can earn and still be relegated.

We used the modeling language ZIMPL 3.3.0 (see [7]) and SCIP 3.0.0, a non-commercial mixed integer programming solver (see [1]) to solve the problem. The optimal solution tells us that a team may earn 57 points and be sixteenth in the league. Indeed, if the top sixteen teams win all of their home games as well as their two away games against the two bottom teams and lose their other games, then each of the first sixteen teams wins 19 games, loses 17 and thus ends the season with 57 points (tie-breakers determine which of these sixteen teams is actually sixteenth). Therefore, to guarantee a fifteenth place, a team needs to earn 58 points, not the widely believed 40.

A General Model

The preceding problem was fairly simple and could have been solved without using an integer program, however it allowed us to introduce the model we will be using. The general model is as follows:

$$\begin{aligned}
\sum_{o \in O} x_g^o &= 1 \quad \forall g \in G \\
\sum_{g \in G} \sum_{\substack{o \in O \\ t \in g}} s_{t,g}^o x_g^o &= p_t \quad \forall t \in T \\
p_{t+1} &\leq p_t \quad \forall t \in T \setminus \{|T|\} \\
x_g^o &\in \{0, 1\} \quad \forall g \in G, o \in O,
\end{aligned}$$

where O is the set of possible outcomes o of a game, and $s_{t,g}^o$ is the number of points given to team $t \in T$ for achieving outcome o in game g . This general model applies to a variety of sports and games; it can be applied in an analogous fashion to any European football league, for instance. We can maximize or minimize the number of points earned by the i th-ranking team for any $i \in \{1, \dots, |T|\}$, answering similar questions such as what is the minimal number of points necessary to qualify for the Champions League or the Europa League in any given National League.

For other sports, depending on the structure of the teams and of the games, some adjustments may be necessary to make the model work; we present some examples in the following sections.

3 Formula 1

The general model can be applied to Formula 1 even though the structure of this sport is very different than that of the Bundesliga. In the history of Formula 1, the scoring system has changed over and over again. In 1994, when Michael Schumacher won his first Formula 1 World Championship, fourteen teams and thus 28 drivers competed in sixteen races. During that year, some drivers changed and thus the overall number of drivers was actually 46. However, for our theoretical analysis, we assume a minimal and constant set of drivers over a season. Points are awarded to the six fastest drivers of a race: 10 to the winner, 6 to the second, 4 to the third, 3 to the fourth, 2 to the fifth and 1 to the sixth.

Applying the model blindly, we let T be the set of drivers, G be the set of races, O be the $|T|!$ different ways of ordering the drivers and $s_{t,g}^o$ be the number of points awarded to driver t in race g for achieving the rank it holds in o . So the model applies, but the number of variables is very large; even if we realize that we only

need to consider the order of the six top drivers, O still has order $\binom{|T|}{6} \cdot 6!$, which is a burdensome 271252800 for $|T| = 28$.

By modifying a bit our way of thinking, we may apply the model in a different way and reduce its size. Let T be the set of drivers, G be the set of games $g = (r, i)$, which consist in finishing i th in race r for $1 \leq i \leq 6$, and $O = T$. In other words, x_g^o with $g = (r, i)$ is 1 if driver o finishes i th in race r and 0 otherwise. Thus the first equation of the model states that exactly one driver finishes i th in race r . The other two equations act as previously explained. However, the model is incomplete; nothing limits the number of positions that a driver can hold in one race. Indeed, with this model, the same driver could finish first, second, third, fourth, fifth and sixth. We must thus add the following constraint

$$\sum_{\substack{g \in G: \\ r \in g}} x_g^o \leq 1 \quad \forall o \in O, r \in R$$

where R is the set of races. This smaller model solves much faster and can be used to compute the optimal solution with respect to different objective functions. For example, $\min p_1$ will minimize the number of points needed to become World Champion, which is 15. We can also calculate

$$16 - \min \sum_{g \in G} x_g^1$$

which determines the maximum number of races in which the World Champion does not finish at least sixth, that is, the maximum number of races for which the World Champion earns no points, which is 14.

By including some additional constraints many other questions can be answered. For instance, if we optimize $\min p_1$ and add the constraint

$$\sum_{\substack{g \in G: \\ g=(r,1)}} x_g^1 = 0,$$

we find out that the minimum number of points that the World Champion can earn while winning no race is 21.

We also learn in case of 24 drivers (or less) that the World Champion must finish at least fifth in some race since $\min p_1$ with the added constraint

$$\sum_{g \in G \setminus \{6\}} x_g^1 = 0$$

is infeasible.

4 The NHL

We now modify the model to examine the National Hockey League. There is a tenacious myth among fans that earning 95 points during the regular season guarantees a team a spot in the playoffs. By making a few adjustments to the general model, we will show that 149 points are needed in the current season structure to ensure such a performance, and 156 points might be needed in the season structure currently under discussion.

The point system is and will remain as follows: two points for a victory, one point for the loss of a game that goes into overtime, and zero points otherwise.

Since 2000, the regular season is structured as follows. The thirty teams are divided equally into two conferences (West and East). Each conference is split into three divisions of five teams. Each team plays eighty-two games during the regular season: six intradivisional games against each of the teams in its own division, four interdivisional intraconference games against each team that is within its conference, but outside its division, and one interconference game against each team in the other conference. Additionally, each team plays an extra interconference game against three teams. Within each conference, eight teams continue to the playoffs: the champion of each division as well as the five teams that fared best among the rest.

Let $C := \{1, 2\}$ be the set of conferences, let $D := \{1, 2, 3\}$ be the set of divisions within a conference, let T be the set of teams $t := (i, d, c)$ which are labeled with $i = 1, \dots, 5$ according to their rank within division d in conference c , let G be the set of games $g := (t_1, t_2, n)$ where n is the number of the game played between teams t_1 and t_2 . Consider the following system:

$$\begin{aligned}
 x_g^{r,1} + x_g^{r,2} + x_g^{o,1} + x_g^{o,2} &= 1 & \forall g \in G \\
 \sum_{\substack{g \in G: \\ g=(t,t',n)}} (2x_g^{r,1} + 2x_g^{o,1} + x_g^{o,2}) + \\
 \sum_{\substack{g \in G: \\ g=(t',t,n)}} (2x_g^{r,2} + 2x_g^{o,2} + x_g^{o,1}) &= p_i^{d,c} & \forall t = (i, d, c) \in T \\
 p_{i+1}^{d,c} &\leq p_i^{d,c} & \forall i \in [4], c \in C, d \in D \\
 x_g^{*,i} &\in \{0, 1\} & \forall g \in G, * \in \{r, o\}, i \in \{1, 2\},
 \end{aligned}$$

where $x_g^{*,j}$ is a binary variable which takes value 1 if game $g := (t_1, t_2, n)$ is won by t_j in $*$ -time (r for regular, o for overtime), and 0 otherwise.

The only difference with the general model is that the third inequality applies to the ranking within a division and not to the global ranking. Note that the extra in-

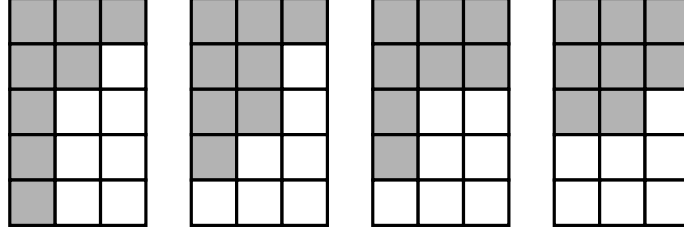


Figure 1: Possible scenarios for playoff qualification.

terconference games cannot be accounted for; we do not know which teams affront each other since those games are not determined by rank. By focusing our attention on a single conference, however, say $c = 1$, we remove the problem since the two conferences access the playoffs independently. In a maximization (resp. minimization) problem, the teams of the selected conference will always win (resp. lose without overtime) all of their interconference games. Accordingly, in the following models we leave out the superscript c indicating the conference for the variables counting the points, since each model only deals with a single conference.

A second problem still arises: we know the ranking of each team in the first conference within its division, but we do not know its overall ranking in the conference. The champion of each division will automatically continue to the playoffs, but we do not know which other five teams will make the cut. We want to find the maximum number of points needed to guarantee a playoff spot, that is, we want to maximize the number of points earned by the fifth-ranked team among the teams of the first conference without the three division champions.

There are eight ranking scenarios to consider up to symmetry, see Figure 1. We represent each team by a square in a matrix-like diagram. Each column represents a division, and each row, the i th position in that division. If the team qualifies for the playoffs, its square is colored gray; otherwise, it is white. We know the top team of each division continues to the playoffs, so these squares will always be gray. We also know that if the i th-ranked team of a division makes it to the playoffs, then any j th-ranked team with $j < i$ in that division will also access the playoffs. Note that the order of the columns is unimportant since we can decide which column is assigned to which division. Besides the top three squares, five more must be gray. There are thus only four diagrams possible.

Clearly, only the lowest gray square of a column may be the last team selected for the playoffs. So in the first diagram, either we maximize p_5^1 and add the constraint that $p_5^1 \leq p_2^2$ or we maximize p_2^2 and add the constraint $p_2^2 \leq p_5^1$. Observe how the relative order of the other teams is unimportant. Thus, for the first scenario, we get the following integer program for the selected conference:

$$\begin{aligned}
\max \quad & p_5^1 + 18 \cdot 2 \\
\text{s.t.} \quad & x_g^{r,1} + x_g^{r,2} + x_g^{o,1} + x_g^{o,2} = 1 \quad \forall g \in G \\
& \sum_{\substack{g \in G': \\ g=(t,t',n)}} (2x_g^{r,1} + 2x_g^{o,1} + x_g^{o,2}) + \\
& \sum_{\substack{g \in G': \\ g=(t',t,n)}} (2x_g^{r,2} + 2x_g^{o,2} + x_g^{o,1}) = p_i^d \quad \forall t := (i,d) \in T \\
& p_{i+1}^d \leq p_i^d \quad \forall i \in [4], d \in D \\
& p_5^1 \leq p_2^2 \\
& x_g^{*,j} \in \{0,1\} \quad \forall g \in G, * \in \{r,o\}, j \in \{1,2\}
\end{aligned}$$

where G' denotes the set of intraconference games and the constant term added in the objective function originates from the interconference games.

We do the same thing for the other diagrams, and we get the following results for the eight scenarios: 148, 148, 149, 149, 148, 148, 148, 148. Thus, to be guaranteed a spot in the playoffs, a team should accumulate 149 points.

Since 2000, the highest number of points collected by a team during the regular season was 124 points, and 95 points weren't enough to qualify for the playoffs only twice (Colorado in 2007 and Dallas in 2010). What can explain such a big gap between reality and theory? Two things stand out as being unrealistic in the solution we found. For one thing, teams in the NHL are mostly of similar strength; each team wins a bit more or a bit less than half of its games. A team winning 67 games is unheard of. Moreover, a relatively small percentage of games end in overtime.

We thus add some constraints to make the model more realistic. Since 2000, the top team had on average 53 wins and the bottom team, 24 wins. Since our model only includes the teams in a single conference, and assuming the probability of winning a game is the same for inter- and intraconference games, we claim that a team wins at most 41 intraconference games and 12 interconference games. Similarly, a team wins at least 18 intraconference games and 6 interconference games:

$$18 \leq \sum_{\substack{g \in G': \\ g=(t,t',n)}} (x_g^{r,1} + x_g^{o,1}) + \sum_{\substack{g \in G: \\ g=(t',t,n)}} (x_g^{r,2} + x_g^{o,2}) \leq 41 \quad \forall t \in T$$

We also add constraints to limit the number of games that go into overtime. Since 2000, about 22% of games have gone into overtime, so we could say that between

19% and 25% of the games played by a team t , that is between 15 and 21 games, go into overtime. Once again, we change this constraint to fit our model better and suppose that a team must have between 12 and 16 intraconference games and between 3 and 5 interconference games go into overtime.

$$12 \leq \sum_{\substack{g \in G': \\ t \in g}} (x_g^{o,1} + x_g^{o,2}) \leq 16 \quad \forall t \in T$$

We can even go further and assume that the likelihood of winning or losing a game that goes into overtime is the same as winning or losing a game in sixty minutes. Thus a team should win between $\frac{24}{82}$ and $\frac{53}{82}$ of its games that go into overtime.

$$\frac{24}{82} \sum_{\substack{g \in G': \\ t \in g}} (x_g^{o,1} + x_g^{o,2}) \leq \sum_{\substack{g \in G': \\ g=(t,t',n)}} x_g^{o,1} + \sum_{\substack{g \in G': \\ g=(t',t,n)}} x_g^{o,2} \leq \frac{53}{82} \sum_{\substack{g \in G': \\ t \in g}} (x_g^{o,1} + x_g^{o,2}) \quad \forall t \in T$$

With these restrictions, the maximum number of points needed to guarantee a place in the playoffs drops to 117 points, which is still much more than 95, but already in a more realistic range.

Now let's consider the restructuration that is currently under discussion in the NHL. The plan is to have two conferences, each divided into two divisions. The two divisions in the West Conference would each be composed of eight teams, whereas the divisions in the East would be formed of seven teams. Each team in the East conference would play six games against each team in its division and forty-six interdivisional and interconference games. In the West conference, the setting would be more complicated: each team would play five games against each team in its division and an additional sixth time with three of those teams, as well as forty-four interdivisional and interconference games. The points would be assigned as before (2 for a win, 1 for a loss with overtime and 0 otherwise). The top four teams of each division would qualify for the playoffs, meaning that each division would access the playoffs independently, and so our model needs only to apply to a single division.

First, we calculate the maximum number of points needed to qualify for the playoffs for a team in the East conference in this structure. Let G'' be the set of intradi-

visional games.

$$\begin{aligned}
& \max \quad p_4 + 46 \cdot 2 \\
& \text{s.t.} \quad x_g^{r,1} + x_g^{r,2} + x_g^{o,1} + x_g^{o,2} = 1 \quad \forall g \in G \\
& \quad \sum_{\substack{g \in G'' : \\ g=(t,t',n)}} (2x_g^{r,1} + 2x_g^{o,1} + x_g^{o,2}) + \\
& \quad \sum_{\substack{g \in G'' : \\ g=(t',t,n)}} (2x_g^{r,2} + 2x_g^{o,2} + x_g^{o,1}) = p_i \quad \forall i \in [7] \\
& \quad p_{i+1} \leq p_i \quad \forall i \in [6] \\
& \quad x_g^{*,j} \in \{0, 1\} \quad \forall g \in G, * \in \{r, o\}, j \in \{1, 2\},
\end{aligned}$$

where p_i is the score of the i th-ranked team in one of the two East divisions. The solution tells us that a team might need 155 points to make the playoffs.

Now let's look at a division in the West conference. The model is exactly the same as before (except with eight teams and forty-four interdivisional and interconference games), but we cannot use the model as is since we do not know which teams will affront each other in the extra intradivisional games.

For now, suppose that every team plays only five times against each other. Then the model can be used and the maximum number of points that the fourth-ranking team can have is 150. What about the extra games? Is it possible for the fourth team to win its three extra games and still be fourth? If so, then $150 + 6 = 156$ would be the number of points needed to guarantee a playoff spot in the West conference. And indeed, this solution is feasible: Consider the extra games represented by an edge between two teams in Figure 2. The top four teams can thus win all of their extra

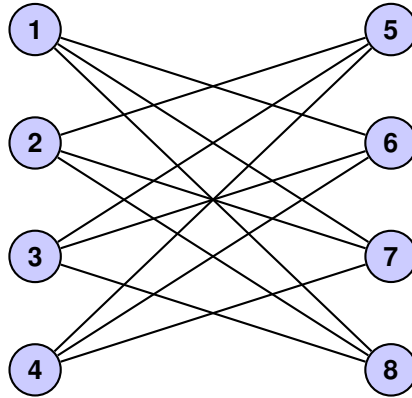


Figure 2: Bipartite graph for extra games of top 4 ranked teams.

games, meaning that the previously fourth-ranked team might have 156 points and still be fourth.

instance	<i>variables</i>	<i>constraints</i>	<i>nodes</i>	<i>time</i>	<i>value</i>
bundesliga	936	341	1	0.13	57
f1_minp1	3164	599	181	19.47	15
f1_maxnopoints	3164	599	91	13.39	14
f1_minp1nowin	3164	600	718	20.10	15
f1_minp1notop5	2712	528	1	0.06	—
nhl-case3	1437	447	1	0.26	149
nhl-real-case3	1929	609	1	0.69	117
nhl-new-east	425	131	1	0.06	155
nhl-new-west	430	138	1	0.02	156

Table 1: Models sizes and running times.

The number of points needed to guarantee a playoff spot will therefore increase from 149 to 155 or 156 (depending on the conference) if the new season structure is adopted. Will this increase be reflected in reality? Moreover, will the slight theoretical difference observed between the East and West conference make it harder for a team in the West conference to reach the playoffs?

5 Results and Conclusion

Table 1 lists the results of the models we discussed using ZIMPL 3.3.0 and SCIP 3.0.0, with SOPLEX 1.7.0 as linear programming (LP) solver, see the SCIP Optimization Suite [10]. All computations were made on a desktop machine with 4 cores (Intel® Xeon® Processor E3-1290 v2 @ 3.7GHz) and 16GB RAM memory. All models and solution files can be obtained from www.zib.de/schlechte/sports-ranking-ip.html under the same names as in Table 1. For each instance, we list the number of *variables* and *constraints* for the corresponding formulation and the needed branch-and-bound *nodes*, the solution *time* in seconds, and in case of feasibility, the optimal *value*.

As can be seen, the integer program we presented can be applied to many different sports to compute within a very short time the minimum or maximum number of points needed to finish *ith* in a competition. Moreover, by adding some additional constraints, the program can determine the number of points needed to ensure a certain achievement, such as avoiding relegation or qualifying for playoffs, even under unusual circumstances, as was seen in the case of Formula One or the NHL. Hence, the presented general model is very powerful and can be easily solved by free and non-commercial software.

We note that the theoretical solutions that we found are always far away from what is observed in practice. Even after adding realistic constraints to the NHL model,

the number of points to guarantee a spot in the playoffs was still much higher than what has been needed in the past. It would be interesting to investigate more deeply which kind of constraints would have to be added to yield solutions even closer to reality.

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