Combining RCC-8 with Qualitative Direction Calculi: Algorithms and Complexity *

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Abstract

Increasing the expressiveness of qualitative spatial calculi is an essential step towards meeting the requirements of applications. This can be achieved by combining existing calculi in a way that we can express spatial information using relations from both calculi. The great challenge is to develop reasoning algorithms that are correct and complete when reasoning over the combined information. Previous work has mainly studied cases where the interaction between the combined calculi was small, or where one of the two calculi was very simple. In this paper we tackle the important combination of topological and directional information for extended spatial objects. We combine some of the best known calculi in qualitative spatial reasoning (QSR), the RCC8 algebra for representing topological information, and the Rectangle Algebra (RA) and the Cardinal Direction Calculus (CDC) for directional information. Although CDC is more expressive than RA, reasoning with CDC is of the same order as reasoning with RA. We show that reasoning with basic RCC8 and basic RA relations is in P, but reasoning with basic RCC8 and basic CDC relations is NP-Complete.

1 Introduction

Qualitative Spatial Reasoning (QSR) is a multi-disciplinary research field that aims at establishing expressive representation formalisms of qualitative spatial knowledge and providing effective reasoning mechanisms. QSR has developed in the last two decades, and dozens of models have been proposed [Cohn and Renz, 2007]. Most of the models focus on single aspect of space, e.g. topology, direction, or shape. This is one major shortcoming of the current QSR formalisms, since many potential applications require multiple aspects.

This paper aims at alleviating this weakness by solving spatial constraints concerning topology and direction, as they are two of the most important aspects of space. More specially, we confine ourselves to the combination of topological and directional constraints over extended planner regions, which are closed, regular and bounded, may have multiple pieces or even holes. We adopt the RCC8 algebra [Randell *et al.*, 1992] to express topological constraints, which is one of the most influential formalism for topological relations. As for directional relations, there are two well-known formalisms that can cope with extended objects, the Rectangle Algebra [Balbiani *et al.*, 1999] and the Cardinal Direction Calculus (CDC) [Goyal and Egenhofer, 2001; Skiadopoulos and Koubarakis, 2005]. RA approximates both the reference and the primary object by their minimum bounding rectangles (MBRs), and relates them by the interval relations between the projected intervals. On the other hand, CDC only approximates the reference object by its MBR, leaving the primary object unchanged. Reasoning with basic CDC constraints is still tractable [Zhang *et al.*, 2008].

Combining spatial constraints of different calculi is a very important problem in QSR. Given a network Θ of topological (RCC8) constraints, and a network Δ of directional (RA or CDC) constraints over the same variables of Θ, the *joint satisfaction problem (JSP)* is to decide when the joint network $\Theta \uplus \Delta$ is satisfiable. Note that we use $\uplus$, instead of $\cup$, to indicate that Θ and Δ involve the same variables.

Since topological and directional information is not independent, $\Theta \uplus \Delta$ may be unsatisfiable despite both Θ and Δ being satisfiable. Solving the joint satisfaction problem is at least as hard as solving the separate satisfaction problem, which means JSP is NP-hard. In this paper, we only consider basic RCC8 constraints and basic RA or CDC constraints. Since arbitrary constraints can always be backtracked to basic constraints, this does not restrict the usefulness of our results.

We show that the JSP over basic RCC8 and basic RA networks can be solved in polynomial time. We also prove that, the JSP over basic RCC8 and basic CDC networks is NP-Complete, by reducing the 3SAT problem to it and devising an exponential decision algorithm.

The rest of this paper proceeds as follows. Section 2 introduces basic notions and related qualitative calculi, including the RCC8 algebra, the Rectangle Algebra (RA), and the Cardinal Direction Calculus (CDC). Sections 3 and 4 consider the computational complexity of the combination of RCC8 with RA and CDC, respectively. Section 5 is the conclusion.

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2 Qualitative calculi

The establishment of a proper qualitative calculus is the key to the success of the qualitative approach to temporal and spatial reasoning. This section introduces basic notions and examples of qualitative calculi.

2.1 Basic notions

Let $\mathbb{U}$ be the universe of temporal or spatial entities, and set $\mathbf{Rel}(\mathbb{U})$ to be the set of binary relations on $\mathbb{U}$. With the usual relational operations of intersection, union, and complement, $\mathbf{Rel}(\mathbb{U})$ is a Boolean algebra. A finite set $\mathcal{B}$ of nonempty relations on $\mathbb{U}$ is *jointly exhaustive and pairwise disjoint* (JEPD for short) if any two entities in $\mathbb{U}$ are related by one and only one relation in $\mathcal{B}$. Write $\langle\mathcal{B}\rangle$ for the subalgebra of $\mathbf{Rel}(\mathbb{U})$ generated by $\mathcal{B}$. Clearly, relations in $\mathcal{B}$ are atoms in the algebra $\langle\mathcal{B}\rangle$. We call $\langle\mathcal{B}\rangle$ a *qualitative calculus* on $\mathbb{U}$, and call relations in $\mathcal{B}$ *basic* relations of the calculus.

A constraint over $\langle\mathcal{B}\rangle$ has the form $x\gamma y$, where γ is a relation in $\langle\mathcal{B}\rangle$. We call $x\gamma y$ a *basic constraint* if γ is a basic relation in $\mathcal{B}$. An important reasoning problem in a qualitative calculus is to determine the *satisfiability* or *consistency* of a network $\Gamma = \{v_i\gamma_{ij}v_j\}_{i,j=1}^n$ of constraints over $\langle\mathcal{B}\rangle$, where Γ is *satisfiable* (or *consistent*) if there is an instantiation $\{a_i\}_{i=1}^n$ in $\mathbb{U}$ such that $(a_i, a_j) \in \gamma_{ij}$ holds for all $1 \leq i, j \leq n$.

Since general constraint networks can be reduced to basic networks by backtracking, we confine ourselves to the consistency of basic networks in this paper.

A basic network $\Gamma = \{v_i\gamma_{ij}v_j\}_{i,j=1}^n$ is *path-consistent* if every subnetwork involving at most three variables is consistent. Path-consistency can be determined in cubic time. For a basic network, it is easy to see that consistency implies path-consistency. The opposite proposition does not always hold, it is true for RCC8 and RA, but not true for CDC.

We use $\mathbf{JSP}(\mathcal{B}_1, \mathcal{B}_2)$ to denote the basic *joint satisfaction problem* over $\mathcal{B}_1$ and $\mathcal{B}_2$. Suppose $\Theta = \{v_i\theta_{ij}v_j\}_{i,j=1}^n$ is a basic constraint network over $\langle\mathcal{B}_1\rangle$, and $\Delta = \{v_i\delta_{ij}v_j\}_{i,j=1}^n$ is a basic constraint network over $\langle\mathcal{B}_2\rangle$ involving the same variables. Then we say $\Theta \uplus \Delta$ is an instance of $\mathbf{JSP}(\mathcal{B}_1, \mathcal{B}_2)$. We say $\Theta \uplus \Delta$ is *bipath-consistent* if Θ and Δ are both path-consistent and $\theta_{ij} \cap \delta_{ij} \neq \varnothing$ for any $1 \leq i, j \leq n$ (recall θ_{ij} and δ_{ij} are subsets of $\mathbb{U} \times \mathbb{U}$). Bipath-consistency can be determined in polynomial time [Gerevini and Renz, 2002].

2.2 RCC8 algebra

A *plane region* (or *region*) is a nonempty regular closed subset of the real plane. In this paper, we only consider bounded regions, which could have multi-pieces and/or have holes. The relations in Table 1 and the converses of **TPP** and **NTPP** form a JEPD set [Randell *et al.*, 1992]. Write $\mathcal{B}_{top}$ for this set, and the RCC8 algebra is generated by $\mathcal{B}_{top}$.

Relation	Symb.	Meaning
equals	**EQ**	$a = b$
disconnected	**DC**	$a \cap b = \varnothing$
externally connected	**EC**	$a \cap b \neq \varnothing \wedge a^\circ \cap b^\circ = \varnothing$
partially overlap	**PO**	$a^\circ \cap b^\circ \neq \varnothing \wedge$ $a \not\subseteq b \wedge a \not\supseteq b$
tangential proper part	**TPP**	$a \subset b \wedge a \not\subset b^\circ$
non-tangential proper part	**NTPP**	$a \subset b^\circ$

Table 1: The set of basic RCC8 relations $\mathcal{B}_{top}$, where a, b are two plane regions and a° and b° are, resp., their interiors.

It is known that, for basic RCC8 networks, path-consistency implies consistency. A cubic realization algorithm was proposed in [Li, 2006]. Since a similar algorithm will be devised later for the combination cases, we give a short description of this algorithm.

Given a basic RCC8 network $\Theta = \{v_i\theta_{ij}v_j\}_{i,j=1}^n$, suppose Θ is path-consistent. An *ntpp-chain* in Θ is defined to be a series of variables $v_{i_1}, v_{i_2}, \cdots, v_{i_k}$ such that $v_{i_s}\mathbf{NTPP}v_{i_{s+1}} \in \Theta$ for all $s = 1, \cdots, k-1$. The *ntpp-level* $l(i)$ of a variable v_i is defined to be the maximum length of the ntpp chains contained in Θ that ends with v_i.

A realization can be constructed as follows. Note that a variable can be interpreted as a bounded region with multiple pieces. We first define for each variable v_i a finite set X_i of *control points* as follows. For each i, introduce a point P_i to v_i; if $v_i\mathbf{EC}v_j$ or $v_i\mathbf{PO}v_j$, then introduce a point P_{ij} to v_i; if $v_i\mathbf{TPP}v_j$ or $v_i\mathbf{NTPP}v_j$, then put all X_i points into X_j. We then expand each point P in X_i a little to obtain a square $s(P)$. These squares are pairwise disjoint. Then, taking the union of these squares, we obtain an instantiation of bounded regions to these v_i. This works for all but the **EC** and **NTPP** constraints. Further modifications are needed to cope with these constraints (cf. [Li, 2006], or Section 3 of this paper).

2.3 Interval Algebra and Rectangle Algebra

The *interval algebra* IA [Allen, 1983] is generated by a set $\mathcal{B}_{int}$ of 13 basic relations between closed intervals (see Table 2). For basic IA constraint networks, path-consistency is sufficient to decide consistency. Moreover, we can construct a canonical solution in the following sense.

Relation	Symb.	Conv.	Meaning
precedes	p	pi	$x^+ < y^-$
meets	m	mi	$x^+ = y^-$
overlaps	o	oi	$x^- < y^- < x^+ < y^+$
starts	s	si	$x^- = y^- < x^+ < y^+$
during	d	di	$y^- < x^- < x^+ < y^+$
finishes	f	fi	$y^- < x^- < x^+ = y^+$
equals	eq	eq	$x^- = y^- < x^+ = y^+$

Table 2: The set of basic interval relations $\mathcal{B}_{int}$, where $x = [x^-, x^+], y = [y^-, y^+]$ are two intervals.

Definition 2.1 (canonical set of intervals). Suppose $\mathfrak{m} = \{[m_i^-, m_i^+]\}_{i=1}^n$ is a set of intervals. Let $E(\mathfrak{m})$ be the set of end points of intervals in $\mathfrak{m}$. We say $\mathfrak{m}$ is *canonical* iff $E(\mathfrak{m}) = [0, M] \cap \mathbb{Z}$, where M is the largest number in $E(\mathfrak{m})$.

For a basic satisfiable IA network, it is not hard to work out the total order of all the end points. Hence we can obtain a canonical solution (by assigning 0 to the first end point, 1 to the second, etc). This gives us the following proposition.

Proposition 2.1. *If a basic IA constraint network is satisfiable, then it has a unique canonical solution.*

IA can be naturally extended to two-dimensional space. We assume an orthogonal basis in the Euclidean plane, and focus on the rectangles sides of which are parallel to the axes of this basis. For a rectangle r, write $I_x(r)$ and $I_y(r)$ as, resp., the x- and y-projection of r. The basic rectangle relation between two rectangles a, b is $\alpha \otimes \beta$ iff $(I_x(a), I_x(b)) \in \alpha$ and $(I_y(a), I_y(b)) \in \beta$, where α, β are two basic IA relations. Apparently, there are $13 \times 13 = 169$ basic rectangle relations, and we write $\mathcal{B}_{rec}$ for this set, i.e., $\mathcal{B}_{rec} = \{\alpha \otimes \beta : \alpha, \beta \in \mathcal{B}_{int}\}$. The Rectangle Algebra (RA) is generated by $\mathcal{B}_{rec}$. The following lemma is straightforward.

Lemma 2.1. *Let $\Delta = \{v_i(\alpha_{ij} \otimes \beta_{ij})v_j\}_{i,j=1}^n$ be a basic RA network. Then Δ is satisfiable iff its projections $\Delta^x = \{x_i \alpha_{ij} x_j\}_{i,j=1}^n$ and $\Delta^y = \{y_i \beta_{ij} y_j\}_{i,j=1}^n$ are satisfiable IA networks.*

We also extend the concept of 'canonical' to RA.

Definition 2.2 (canonical set of rectangles). A set of rectangles $\{m_i\}_{i=1}^n$ is *canonical* iff its x- and y-projections, $\{I_x(m_i)\}_{i=1}^n$ and $\{I_y(m_i)\}_{i=1}^n$ are canonical sets of intervals.

From Prop. 2.1, clearly each path-consistent basic RA network has a unique canonical solution.

The RA relations can be extended from rectangles to arbitrary bounded regions. For a bounded region b, its *minimum bounding rectangle* (MBR), denoted by $\mathcal{M}(b)$, is defined to be the smallest rectangle containing b. The extended RA relation between two bounded regions a and b is defined to be the RA relation between $\mathcal{M}(a)$ and $\mathcal{M}(b)$. These extended RA relations are useful for expressing directional information.

2.4 Cardinal Direction Calculus

Given a bounded region b in the real plane, by extending the four edges of $\mathcal{M}(b)$, we partition the plane into nine tiles, denoted by b^{ij} ($1 \le i, j \le 3$), see Fig. 1 (left) for illustration.

For a primary region a and a reference region b, the CDC relation of a to b, denoted by δ_{ab}, is encoded in a 3×3 Boolean matrix $(d_{ij})_{1 \le i,j \le 3}$, where $d_{ij} = 1$ iff $a^\circ \cap b^{ij} \neq \varnothing$ (here a° is the interior of a). For example, in Fig. 1 we have $\delta_{ab} = \delta_{a'b'} = \begin{bmatrix} 0&0&0\\1&0&0\\0&0&0 \end{bmatrix}$, $\delta_{ba} = \delta_{b'a'} = \begin{bmatrix} 0&0&1\\0&0&1\\0&0&1 \end{bmatrix}$.

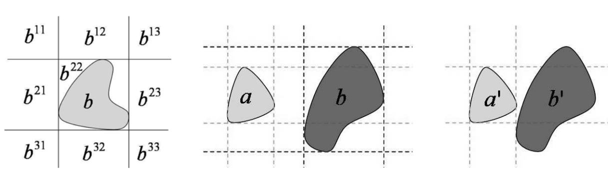

Figure 1: Illustrations of the Cardinal Direction Calculus

A CDC relation could be any but the zero Boolean matrix, so there are $2^9 - 1 = 511$ basic relations in CDC. We denote this set by $\mathcal{B}_{cdc}$. Since only the reference object is approximated, CDC is in general more expressive than the RA.

Definition 2.3. For a pair of basic CDC relation (δ, γ), the *x-projective interval relation* of (δ, γ), written $\iota^x(\delta, \gamma)$, is defined to be the union of all IA basic relation α s.t. $(I_x(m_1), I_x(m_2)) \in \alpha$, where $\{m_1, m_2\}$ is a solution of $\{v_1 \delta v_2, v_2 \gamma v_1\}$.

If $\{v_1 \delta v_2, v_2 \gamma v_1\}$ is consistent, we can prove that $\iota^x(\delta, \gamma)$ is either a basic IA relation in $\{\mathsf{o}, \mathsf{s}, \mathsf{d}, \mathsf{f}, \mathsf{eq}, \mathsf{oi}, \mathsf{si}, \mathsf{di}, \mathsf{fi}\}$ or a non-basic IA relation in $\{\mathsf{p} \cup \mathsf{m}, \mathsf{pi} \cup \mathsf{mi}\}$. A similar definition and result holds for the y-direction. For example, from Fig. 1, we have $\iota^x(\delta_{ab}, \delta_{ba}) = \mathsf{p} \cup \mathsf{m}$, $\iota^y(\delta_{ab}, \delta_{ba}) = \mathsf{d}$.

We call $\iota(\delta, \gamma) = \iota^x(\delta, \gamma) \otimes \iota^y(\delta, \gamma)$ the RA relation *induced by* (δ, γ). In general, for a basic CDC constraint network $\Delta = \{v_i \delta_{ij} v_j\}_{i,j=1}^n$, we call $\iota(\Delta) = \{v_i \iota_{ij} v_j\}_{i,j=1}^n$ the *induced RA constraint network* of Δ, where $\iota_{ij} = \iota(\delta_{ij}, \delta_{ji})$.

We next consider special solutions of CDC networks.

Definition 2.4 (maximal solution). A solution $\{m_i\}_{i=1}^n$ of a CDC constraint network Δ is *maximal* if $m_i' \subseteq m_i$ holds for any solution $\{m_i'\}_{i=1}^n$ of Δ with $\mathcal{M}(m_i) = \mathcal{M}(m_i')$.

Definition 2.5 (regular solution). A solution $\mathfrak{m} = \{m_i\}_{i=1}^n$ of a CDC constraint network is *regular* if $\mathfrak{m}$ is maximal and $\{\mathcal{M}(m_i)\}_{i=1}^n$ is a canonical set of rectangles.

Although a basic CDC network may have more than one regular solutions, we have the following lemma.

Lemma 2.2. *Suppose Δ is a basic CDC network with a solution $\{m_i\}_{i=1}^n$. Then there exists a unique regular solution $\{m_i'\}_{i=1}^n$ of Δ s.t. for any i, j the basic RA relation between m_i' and m_j' is the same as that between m_i and m_j.*

[Zhang *et al.*, 2008] devised a cubic algorithm which determines whether a basic CDC network has a (possibly disconnected) solution, and gives a regular one if possible. Here we review the algorithm as it will be used in Section 4. First, we compute a canonical solution of the induced (possibly non-basic) RA network, which yields a grid space. Next, we remove the grids that violate some constraints from each rectangle. Third, we check whether what we have obtained is a valid solution. In the following, we give a detailed description with a running example illustrated in Table 3 and Fig. 2.

	δ_{ij}	δ_{ji}	$\iota_{ij}^x \otimes \iota_{ij}^y$	$\rho_{ij}^x \otimes \rho_{ij}^y$
$(1,2)$	$\begin{bmatrix} 0&0&0\\1&1&0\\1&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&1&1\\0&1&1\\0&0&0 \end{bmatrix}$	$\mathsf{o} \otimes \mathsf{o}$	$\mathsf{o} \otimes \mathsf{o}$
$(1,3)$	$\begin{bmatrix} 0&0&0\\1&0&0\\1&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}$	$(\mathsf{m} \cup \mathsf{p}) \otimes \mathsf{fi}$	$\mathsf{p} \otimes \mathsf{fi}$
$(2,3)$	$\begin{bmatrix} 1&1&0\\1&0&0\\0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0\\0&0&1\\0&1&1 \end{bmatrix}$	$\mathsf{o} \otimes \mathsf{oi}$	$\mathsf{o} \otimes \mathsf{oi}$

Table 3: example of solving a basic CDC network

Step 1. Compute the induced RA network Γ^0 of Δ.
Step 2. Refine Γ^0 to a basic RA network $\Gamma = \{v_i(\rho_{ij}^x \otimes \rho_{ij}^y)v_j\}_{i,j=1}^n$ by setting $\rho_{ij}^x = \iota_{ij}^x \backslash (\mathsf{m} \cup \mathsf{mi})$ and $\rho_{ij}^y = \iota_{ij}^y \backslash (\mathsf{m} \cup \mathsf{mi})$. If Γ is unsatisfiable, then it can be proven that neither is Δ. Suppose Γ is satisfiable and calculate its canonical solution $\mathfrak{m}^\Gamma = \{m_i^\Gamma\}_{i=1}^n$ (cf. Fig. 2 left).

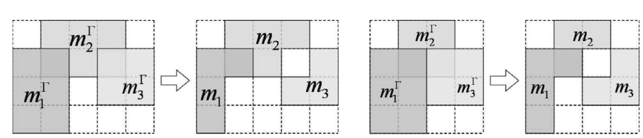

Figure 2: Solution $\mathfrak{m}$ derived from a canonical solution $\mathfrak{m}^\Gamma$.

Step 3. This step tries to find a solution $\mathfrak{m} = \{m_i\}_{i=1}^n$ s.t. $\mathcal{M}(m_i) = \mathcal{M}(m_i^\Gamma)$. Recall a basic CDC relation δ_{ij} is represented as a 3×3 Boolean matrix $((\delta_{ij})_{xy})$. If $\mathfrak{m}$ is a solution, $m_i^\circ \cap (m_j)^{xy} = \varnothing$ holds for any $(\delta_{ij})_{xy} = 0$, where $(m_j)^{xy}$ is one of the nine tiles generated by $\mathcal{M}(m_j)$ (cf. Fig. 1). This means, to make $\mathfrak{m}$ a solution to Δ, we need to exclude all impossible grids from m_i^Γ. Set $T_i = \bigcup\{(m_j^\Gamma)^{xy} : (\delta_{ij})_{xy} = 0\}_{j=1}^n$. Let m_i be the closure of $m_i^\Gamma \setminus T_i$ (cf. Fig. 2 left).
Step 4. Note we may have removed too many tiles so that some constraint like $(\delta_{ij})_{xy} = 1$ is violated. The last step then checks whether $\mathfrak{m} = \{m_i\}_{i=1}^n$ is a solution of Δ. If $\mathfrak{m}$ is not, we can prove that Δ has no solution at all. Otherwise, it's not difficult to find out that $\mathfrak{m}$ is a regular solution.

We need to point out that other regular solutions may exist (cf. Fig. 2 right). We can get all of them by repeating Step 2 to 4 using every possible refinement of Γ^0.

3 Combination of RCC8 and RA networks

We now consider the combination of RCC8 and RA. First we show that bipath-consistency is not sufficient for consistency in $\mathbf{JSP}(\mathcal{B}_{top}, \mathcal{B}_{rec})$. Let $\Gamma = \{v_i \gamma_{ij} v_j\}_{i,j=1}^4$ be the basic RA network as specified in Fig. 3, where $\mathfrak{m}^\Gamma = \{m_i^\Gamma\}_{i=1}^4$ is the canonical solution of Γ. Let $\Theta = \{v_i \theta_{ij} v_j\}_{i,j=1}^4$ be the RCC8 network where $\theta_{12} = \theta_{34} = \mathbf{EC}$ and $\theta_{13} = \theta_{24} = \mathbf{DC}$. Although we can verify $\Theta \uplus \Gamma$ is bipath-consistent, it is not satisfiable. Otherwise, for any solution $\{m_i\}_{i=1}^4$, we have $\mathcal{M}(m_1) \cap \mathcal{M}(m_2) = \mathcal{M}(m_3) \cap \mathcal{M}(m_4)$ is a singleton, say P. As $\theta_{12} = \theta_{34} = \mathbf{EC}$, we know $P \in m_i$ $(i = 1, 2, 3, 4)$, which contradicts $\theta_{13} = \mathbf{DC}$.

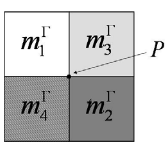

Figure 3: A bipath-consistent but inconsistent network Γ

We call point P in the above configuration a *conflict* point.

Definition 3.1 (conflict point). Let Θ be a basic RCC8 network and Γ a basic RA network. Suppose $\mathfrak{m}^\Gamma$ is the canonical solution of Γ. A point Q is called a *conflict point* of m_i^Γ if there exists j such that $m_i^\Gamma \cap m_j^\Gamma = \{Q\}$ and $\theta_{ij} = \mathbf{EC}$. We write C_i for the set of all conflict points of m_i^Γ.

Clearly, each conflict point of m_i^Γ is also a corner point of m_i^Γ. This implies that m_i^Γ and m_j^Γ may have at most one common conflict point. Moreover, if $\mathfrak{m} = \{m_i\}_{i=1}^n$ is a solution of $\Theta \uplus \Gamma$ s.t. $\mathcal{M}(m_i) = m_i^\Gamma$ for all $1 \leq i \leq n$, then $Q \in m_i$. This means $C_i \subset m_i$. As a consequence, we have

$$\theta_{ij} = \mathbf{DC} \Rightarrow C_i \cap C_j = \varnothing \quad (1 \leq i, j \leq n) \qquad (1)$$

The following theorem shows that this is also sufficient.

Theorem 3.1. *Suppose $\Theta \uplus \Gamma$ is bipath-consistent. Then $\Theta \uplus \Gamma$ is satisfiable iff Equation 1 holds.*

Proof. We only need to show the 'only if' part. Similar to the method for RCC8 alone (cf. Section 2.3 and [Li, 2006]), assuming the ntpp-level $l(i)$ has been computed, we construct a solution $\mathfrak{m} = \{m_i\}_{i=1}^n$ which also satisfies $\mathcal{M}(m_i) = m_i^\Gamma$.
Step 1. Selection of control points

For each v_i we select a set of control points X_i. First select one point from each edge of m_i^Γ and for X_i. Then, for any $j > i$ with $\theta_{ij} = \mathbf{EC}$ or $\mathbf{PO}$, select a point P_{ij} from $m_i^\Gamma \cap m_j^\Gamma$ (which is nonempty because of the bipath-consistency of $\Theta \uplus \Gamma$) for both X_i and X_j. Note that $m_i^\Gamma \cap m_j^\Gamma$ could either be a single point, or a line segment, or a rectangle. When choosing P_{ij} from $m_i^\Gamma \cap m_j^\Gamma$, we should avoid the special points as much as possible in order to make Step 3 simpler. Moreover, the points are required to be distinct if possible. We write $\mathcal{P}$ for the set of all the control points.
Step 2. Basic regions associated to control points

For each control point Q, we construct a series of sectors $\{q^{i,k} : k = 1, \cdots, 4\}_{i=1}^n$ and a series of squares $\{q^{(i)}\}_{i=1}^n$ (see Fig. 4). We call them the basic regions associated to Q. Note that we use an upper case letter to denote a control point, and the corresponding lower case letter (with indices) to denote basic regions. The sectors are chosen in such way as it allows us to distinguish up to four connecting regions in cases where Q is a corner point (such as point P in Fig.3).

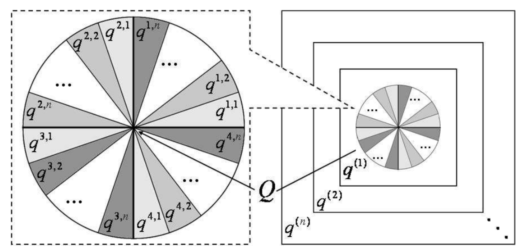

Figure 4: Basic regions of a control point Q

For any two different control points, we require their outermost squares to be disjoint. Furthermore, a basic region must be small enough so that it is not crossed by the border of any m_i^Γ of which Q is not a boundary point.
Step 3. Region construction

For each control point Q, set $q^i = \bigcup_{k=1}^4 q^{i,k}$. Let

$$\begin{aligned} a_i^1 &= m_i^\Gamma \cap \bigcup\{q^i : Q \in X_i\} \\ a_i^2 &= a_i^1 \cup (m_i^\Gamma \cap \bigcup\{q^j : \theta_{ij} = \mathbf{PO}, Q \in X_i \cap X_j\}) \\ a_i^3 &= a_i^2 \cup \bigcup\{a_j^2 : \theta_{ji} = \mathbf{TPP} \text{ or } \theta_{ji} = \mathbf{NTPP}\} \\ a_i^4 &= a_i^3 \cup \bigcup\{q^{(l(i))} : \theta_{ji} = \mathbf{NTPP}, Q \in a_j^3\} \end{aligned}$$

Set $m_i = a_i^4$ and $\mathfrak{m} = \{m_i\}_{i=1}^n$. It is easy to prove that $\mathfrak{m}$ satisfies all RCC8 constraints in Θ. To show $\mathfrak{m}$ also satisfies Γ, we need only prove $\mathcal{M}(m_i) = m_i^\Gamma$ for each i. It is clear that a_i^1 and a_i^2 are subsets of m_i^Γ. By the choice of X_i, we know $m_i^\Gamma = \mathcal{M}(a_i^1) = \mathcal{M}(a_i^2)$. If $\theta_{ji} = \mathbf{TPP}$ or $\mathbf{NTPP}$, then $m_j^\Gamma \subseteq m_i^\Gamma$ by bipath-consistency. This implies $\mathcal{M}(a_i^3) = m_i^\Gamma$. Furthermore, if $\theta_{ji} = \mathbf{NTPP}$, we have $(m_j^\Gamma, m_i^\Gamma) \in \mathsf{d} \otimes \mathsf{d}$ by bipath-consistency. So for any control point Q in $a_j^3 \subseteq m_j^\Gamma$, Q is also in the interior of m_i^Γ. Therefore, by the choice of basic regions, we know the outmost square $q^{(n)}$ at Q, hence $q^{(l(i))}$, is contained in m_i^Γ. Therefore, $\mathcal{M}(a_i^4) = m_i^\Gamma$. This proves that $\mathfrak{m}$ is a solution to $\Theta \uplus \Gamma$. □

As a corollary, we have **JSP**$(\mathcal{B}_{top}, \mathcal{B}_{rec})$ is in P.

Corollary 3.1. *For a basic RCC8 network Θ and a basic RA network Γ, consistency of $\Theta \uplus \Gamma$ can be decided in cubic time.*

Proof. Bipath-consistency of $\Theta \uplus \Gamma$ can be checked in cubic time. We can construct the unique canonical rectangle solution of Γ in cubic time. The conflict point set C_i can also be computed in cubic time. That is, the condition of Theorem 3.1 can be checked in cubic time. □

4 Combination of RCC8 and CDC networks

This section discusses the joint satisfaction problem of a basic RCC8 network and a basic CDC network. We first show **JSP**$(\mathcal{B}_{top}, \mathcal{B}_{cdc})$ is NP-hard, and then show it is also in NP.

Let $\Theta = \{v_i \theta_{ij} v_j\}_{i,j=1}^n$ be a basic RCC8 network and $\Delta = \{v_i \delta_{ij} v_j\}_{i,j=1}^n$ a basic CDC network. Suppose $\Theta \uplus \Delta$ is bipath-consistent, and let $\mathfrak{m}^\Delta = \{m_i^\Delta\}_{i=1}^n$ be one regular solution of Δ. Clearly, m_i^Δ and m_j^Δ may meet at two or more but finite points (see Fig. 2 for example). These points may introduce conflicts when combined with topological constraints. This is different from RA, where two variables may have at most one conflict point. This will essentially increase the computational complexity.

Lemma 4.1. *There is a polynomial reduction from 3SAT to **JSP**$(\mathcal{B}_{top}, \mathcal{B}_{cdc})$.*

Proof. For each 3SAT instance $\varphi = \bigwedge_{i=1}^m (q_{i1} \vee q_{i2} \vee q_{i3})$ with n propositional variables $\{p_k\}_{k=1}^n$, where $q_{ij} \in \{p_k\}_{k=1}^n \cup \{\neg p_k\}_{k=1}^n$, we construct a **JSP**$(\mathcal{B}_{top}, \mathcal{B}_{cdc})$ instance $\Theta_\varphi \uplus \Delta_\varphi$ such that φ is satisfiable iff $\Theta_\varphi \uplus \Delta_\varphi$ is consistent. There are three types of spatial variables in Θ_φ and Δ_φ.

Step 1. Grid variables

We introduce $10n$ spatial variables G_{ij} as grid variables, where $1 \leq i \leq 2n$ and $1 \leq j \leq 5$. The reason we call them 'grid variables' will become clear soon. The CDC constraints between them are specified as in Fig. 5 (left). The RCC8 constraint between $G_{ij}, G_{i'j'}$ is $\mathbf{EC}$ if they are 4-neighbors (i.e. $\{|i-i'|, |j-j'|\} = \{0,1\}$), otherwise $\mathbf{DC}$. These $\mathbf{EC}$ constraints make sure that there is no gap between the MBRs of two neighboring grid variables.

Grid variables are mainly used to locate other spatial variables. For a new variable v and a grid variable G_{ij}, we say v *occupies* G_{ij} if $v \cap \mathcal{M}(G_{ij})$ is nonempty, and its MBR is $\mathcal{M}(G_{ij})$, i.e. $\mathcal{M}(v \cap \mathcal{M}(G_{ij})) = \mathcal{M}(G_{ij})$.

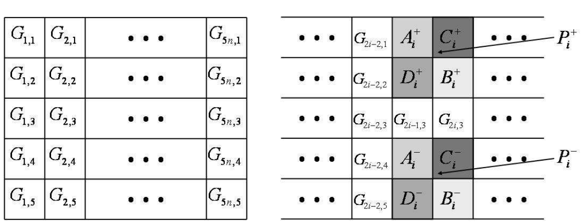

Figure 5: Grids; spatial variables for propositional variables

Step 2. Spatial variables for propositional variables

For each propositional variable p_i in φ, four spatial variables A_i, B_i, C_i and D_i are introduced. Take A_i as example. By assigning the CDC constraints between A_i and the grid variables, we require A_i occupies $G_{2i-1,1}$ and $G_{2i-1,4}$, but has an empty intersection with the interior of the MBR of any other grid variable, see Fig. 5 (right) for illustration, where A_i^+ and A_i^- are used to denote the two parts of the sketch of A_i. Easy to see, $A_i \cap B_i$ contains at most two points, viz. P_i^+ and P_i^-, and so does $C_i \cap D_i$.

As for the topological constraints, we set $\theta_{A_j,B_j} = \theta_{C_j,D_j} = \mathbf{EC}$, and all the others to be $\mathbf{DC}$. The $\mathbf{EC}$ constraints imply that both $A_i \cap B_i$ and $C_i \cap D_i$ are nonempty. On the other hand, since $A_i \mathbf{DC} C_i$, we can get the conclusion that A_i and B_i must share only one of P_i^+ and P_i^-, while C_i and D_i share the other one.

Step 3. Spatial variables for clauses

For each clause $(q_{j1} \vee q_{j2} \vee q_{j3})$ in φ, two new spatial variables E_j and F_j are introduced, both of which occupy three grids. The precise occupied grids are set according to the variables and signs of q_{jk}. One example is given in Fig. 6 to illustrate the construction, where we assume $q_{j1} = p_{i_1}, q_{j2} = \neg p_{i_2}, q_{j3} = \neg p_{i_3}$. As for the topological constraints, we set $\theta_{E_j,F_j} = \mathbf{EC}$, while all others (e.g., θ_{A_j,E_i}) are $\mathbf{DC}$. This implies $E_j \cap F_j$ contains at least one point of $P_{i_1}^-, P_{i_2}^+, P_{i_3}^+$. We assert that it won't be the case that $A_{i_1} \cap B_{i_1} = \{P_{i_1}^-\}$, $A_{i_2} \cap B_{i_2} = \{P_{i_2}^+\}$ and $A_{i_3} \cap B_{i_3} = \{P_{i_3}^+\}$. Otherwise, some $\mathbf{DC}$ constraint, e.g. that between A_{i_1} and E_j, will be violated.

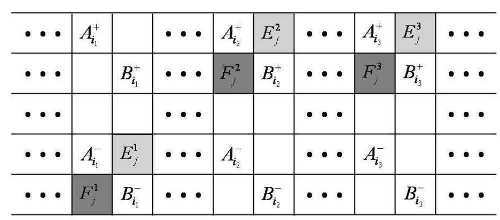

Figure 6: Spatial variables for clauses

So far we have finished the construction. Next, we show that φ is satisfiable iff $\Theta_\varphi \uplus \Delta_\varphi$ is consistent.

If $\Theta_\varphi \uplus \Delta_\varphi$ has a solution $\mathfrak{m}$, we can obtain an assignment $\pi : \{p_i\}_{i=1}^n \to \{true, false\}$ s.t. $\pi(p_i) = true$ iff $A_i \cap B_i = \{P_i^+\}$ by $\mathfrak{m}$. By Step 3, we can verify that π satisfies φ. On the other side, if π is an assignment that satisfies φ, we show $\Theta_\varphi \uplus \Delta_\varphi$ is consistent. The idea is to introduce an instance of **JSP**$(\mathcal{B}_{top}, \mathcal{B}_{rec})$, in which we have two spatial variables A_i^+ and A_i^- instead of A_i (also for B_i, C_i, D_i), and three variables E_j^k $(1 \leq k \leq 3)$ instead of E_j (also for F_j). The RA constraints are set according to Fig. 5 and Fig. 6, while the RCC8 constraints are set by Θ_φ and π. It can be proved that this new joint network satisfies Eq. 1, and a solution can be obtained in cubic time. A solution of $\Theta_\varphi \uplus \Delta_\varphi$ can then be obtained by merging the related regions (e.g. merging A_i^+ and A_i^- into A_i). The verification is straightforward. □

In this way we proved that 3SAT can be reduced to **JSP**$(\mathcal{B}_{top}, \mathcal{B}_{cdc})$. The following theorem is clear.

Theorem 4.1. ***JSP***$(\mathcal{B}_{top}, \mathcal{B}_{cdc})$ *is NP-hard.*

In the remainder of this section, we show **JSP**$(\mathcal{B}_{top}, \mathcal{B}_{cdc})$ is also in NP. Let $\Theta \uplus \Delta$ be an instance of **JSP**$(\mathcal{B}_{top}, \mathcal{B}_{cdc})$. If $\Theta \uplus \Delta$ is consistent, then it can be proved that Δ has a regular solution (cf. Dfn. 2.5) $\mathfrak{m}^\Delta = \{m_i^\Delta\}_{i=1}^n$ and $\Theta \uplus \Delta$ has a solution $\mathfrak{m} = \{m_i\}_{i=1}^n$ such that

- $m_i \subseteq m_i^\Delta$ for each i;
- $m_i \cap \mathcal{M}(g) \neq \varnothing$ for each grid $g \subseteq m_i^\Delta$ and each i.

If this is the case, we call $\mathfrak{m}^\Delta$ a *consistent regular solution*, and say $\mathfrak{m}$ is consistent with $\mathfrak{m}^\Delta$. Therefore, to determine whether $\Theta \uplus \Delta$ is consistent, we need only check if there exists a consistent regular solution.

Suppose $\mathfrak{m}^\Delta$ is a regular solution of Δ. Set

$$U_i = \{\text{all end points of } m_i^\Delta\} \quad (2)$$

$$N_i = \bigcup\{\partial m_i^\Delta \cap \partial m_j^\Delta : \theta_{ij} = \mathbf{NTPP}\}, \quad (3)$$

where ∂A is the boundary of A. Set $U = \bigcup_{i=1}^n U_i$. A function f from $\mathcal{I} = \{1, 2, \cdots, n\}$ to $\mathcal{P}(U)$, i.e. the power set of U, is called a (conflict point) selection function if for any $1 \leq i, j \leq n$

$$f(i) \subseteq U_i \setminus N_i$$

$$\theta_{ij} = \mathbf{DC} \Rightarrow f(i) \cap f(j) = \varnothing$$

$$\theta_{ij} = \mathbf{EC} \;\&\; m_i^\Delta \cap m_j^\Delta \text{ is finite} \Rightarrow f(i) \cap f(j) \neq \varnothing$$

A consistent regular solution $\mathfrak{m}^\Delta$ satisfies:

- $\mathsf{edge}_k(\mathcal{M}(m_i^\Delta)) \cap m_i^\Delta \not\subseteq N_i$;
- if $\theta_{ij} = \mathbf{PO}$ then $(m_i^\Delta \cap m_j^\Delta)^\circ \neq \varnothing$;
- there exists a selection function $f : \mathcal{I} \to \mathcal{P}(U)$,

where $\mathsf{edge}_k(r)$ returns the k-th edge of a rectangle r ($1 \leq k \leq 4$). The first two conditions are easy to check. As for the third, suppose $\mathfrak{m}$ is a solution of $\Theta \uplus \Delta$ that is consistent with $\mathfrak{m}^\Delta$. Define $f(i) = m_i \cap U_i$. It is straightforward to check that f is a selection function.

On the other hand, suppose $\mathfrak{m}^\Delta$ is a regular solution of Δ that satisfies the above three conditions. Then we can construct a solution of $\Theta \uplus \Delta$ that is consistent with $\mathfrak{m}^\Delta$. This process is similar to that used in **JSP**$(\mathcal{B}_{top}, \mathcal{B}_{rec})$. We include each point in $f(i)$ into the control points set X_i. Besides, for every variable v_i, we select a control point from each grid in m_i^Δ. The process is polynomial. Note that the cardinality of regular solutions of Δ and the cardinality of all possible selection functions f are of order 2^n. We have

Theorem 4.2. ***JSP***$(\mathcal{B}_{top}, \mathcal{B}_{cdc})$ *is in NP.*

Together with the NP-hard result, we have

Theorem 4.3. ***JSP***$(\mathcal{B}_{top}, \mathcal{B}_{cdc})$ *is NP-complete.*

5 Conclusion

In this paper, we analyzed the interaction between topological and directional constraints for extended spatial regions. We used the rectangle algebra RA and CDC to describe direction information, and RCC8 to express topological constraints. We have shown that the problem of deciding consistency of a joint basic network of RCC8 and RA constraints is still in P, while the one for RCC8 and CDC is NP-Complete.

Our results represent a large step towards the applicability of qualitative spatial reasoning techniques for real-world problems. In particular the result about RCC8 and RA is very promising as it enables efficient reasoning about these two important calculi. It also means that if efficient reasoning is important for a potential application, developers should aim for representing direction information using RA rather than using CDC. Our results about combining RCC8 and CDC are important from a theoretical point of view as they are the first formal results for this combination.

Some related results have been obtained in [Li, 2007], where RCC8 was combined with DIR9, which is a very small sub-algebra of RA that contains only nine basic relations. For a set of directional constraints over DIR9 and a set of topological constraints over a maximal tractable subclass of RCC8, [Li, 2007] proved that the joint satisfaction problem can be reduced to two independent satisfaction problems over RCC8 and RA, respectively. It is worth noting that this work does not restrict constraints to basic ones. While [Li, 2007] shows that reasoning with RCC8 and DIR9 is decidable, it is the present work which proves the decidability of reasoning with RCC8 and the whole RA. Our future work will extend results obtained here to large tractable subclasses of RA and RCC8.

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